TECHNICAL REPORT

Fundamental lemmas for the determination of optimal control strategies for a class of single machine family scheduling problems

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Abstract

Four lemmas, which constitute the theoretical foundation necessary to determine optimal control strategies for a class of single machine family scheduling problems, are presented in this technical report. The scheduling problem is characterized by the presence of sequence-dependent batch setup and controllable processing times; moreover, the generalized due-date model is adopted in the problem. The lemmas are employed within a constructive procedure (proposed by the Author and based on the application of dynamic programming) that allows determining the decisions which optimally solve the scheduling problem as functions of the system state. Two complete examples of single machine family scheduling problem are included in the technical report with the aim of illustrating the application of the fundamental lemmas in the proposed approach.

1 Introduction

In [1], a class of single machine family scheduling problems (mainly characterized by multiclass jobs, generalized due-dates, and controllable processing times) has been formalized as an optimal control problem. Its solution consists of optimal control strategies which are functions of the system state, and therefore they are able to provide the optimal decisions for any actual machine behavior (the single machine is assumed to be unreliable and then perturbations, such as breakdowns, generic unavailabilities, and slowdowns, may affect the nominal behavior of the system). However, the scheduling problem in [1] has been solved under the assumption that, for each class of jobs, any unitary tardiness cost is greater than the unitary cost related to the deviation from the nominal service time. In order to remove such a strong hypothesis and to extend the scheduling model by adding setup times and, especially, setup costs, new fundamental lemmas have been defined. They are employed within the constructive procedure proposed in [2] that solves, from a control-theoretic perspective, a single machine scheduling problem with sequence-dependent batch setup and controllable processing times.

This technical report is organized as follows. Some preliminary definitions are reported in section 2. The four new lemmas are presented in sections 3, together with their complete proofs. Nine numerical examples aiming at illustrating how lemmas 1 and 2 work are in section 4. Finally, sections 5 and 6 present two complete example which explain the application of the procedure proposed in [2] to two single machine family scheduling problems (the latter with setup).

2 Definitions

Definition 1. Consider a function $f(x)$ which is continuous, nondecreasing, and piece-wise linear function of the independent variable x. Let $f(x)$ be characterized by $M \geq 1$ changes of slopes and let γ_i, $i = 1, \ldots, M$, be the values of the horizontal axis at which the slope changes ($\gamma_{i+1} > \gamma_i$, $\forall\, i = 1, \ldots, M-1$). In this connection, let μ_0 be the slope in interval $(-\infty, \gamma_1)$, μ_i be the slope in interval $[\gamma_i, \gamma_{i+1})$, $i = 1, \ldots, M-1$, and μ_M be the slope in interval $[\gamma_M, +\infty)$ ($\mu_{i+1} \neq \mu_i$, $\forall\, i = 0, \ldots, M-1$). Moreover, it is assumed $f(x) = 0$ for any $x \leq \gamma_1$; then, $\mu_0 = 0$. An example of function $f(x)$ following this definition is in figure 1.

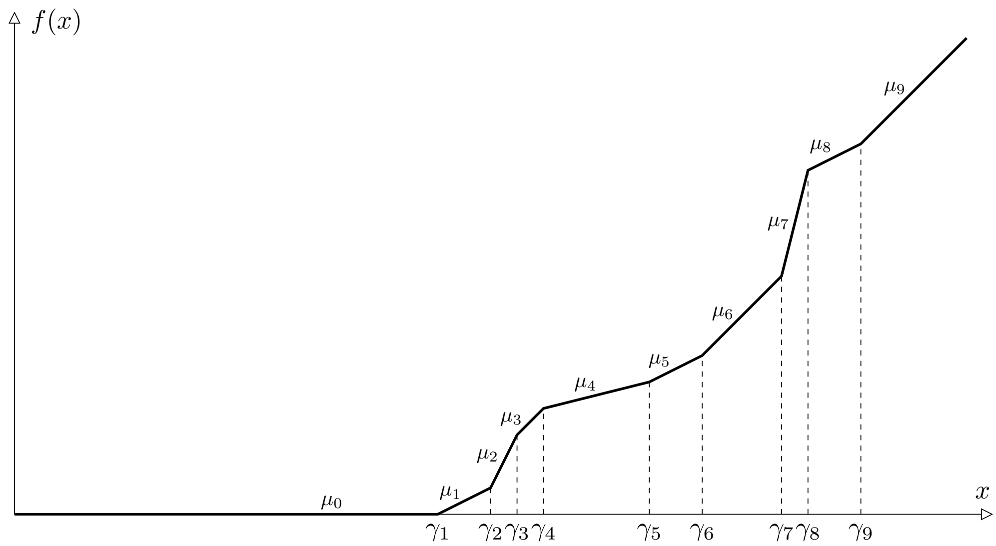


Figure 1: Example of function $f(x)$.

Definition 2. With reference to $f(x)$, as defined by definition 1, let $f(x+t)$ be a continuous, nondecreasing, and piece-wise linear function of the independent variable x, parameterized by the real value t. An example of function $f(x+t)$ following this definition is in figure 2.

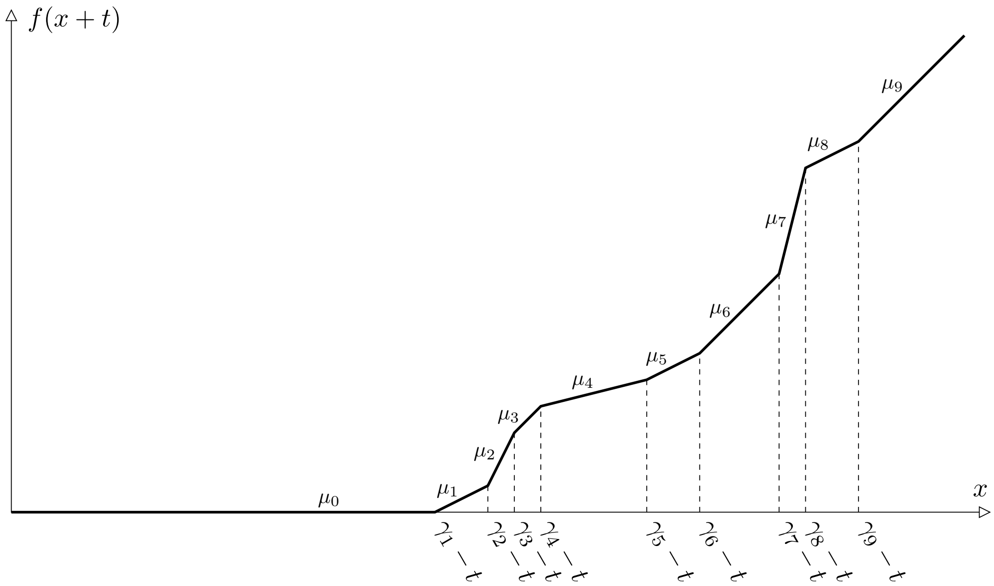


Figure 2: Example of function $f(x+t)$.

Definition 3. Consider a function $g(x)$ which is noncontinuous, nonincreasing, and piece-wise linear function of the independent variable x. Let $g(x)$ be defined as

$$g(x) = \begin{cases} -\nu(x - x_2) & x \in [x_1, x_2) \\ 0 & x \notin [x_1, x_2) \end{cases} \tag{1}$$

An example of function $g(x)$ following this definition is in figure 3.

Figure 3: Example of function $g(x)$.

3 Lemmas

In connection with functions $f(x)$ and $g(x)$ as defined by definitions 1 and 3, let:

- A be the set of indices i, $i \in \{1, \dots, M\}$, such that $\mu_{i-1} < \nu$ and $\mu_i \geq \nu$; in this connection, let $|A|$ be the cardinality of set A and, if $|A| > 0$, let a_j, $j = 1, \dots, |A|$, be the generic element of set A; thus, γ_{a_j}, $j = 1, \dots, |A|$, are the value of the horizontal axis at which the slope of $f(x)$ changes from a value less than ν to a value greater than or equal to ν;
- B be the set of indices i, $i \in \{1, \dots, M\}$, such that $\mu_{i-1} \geq \nu$ and $\mu_i < \nu$; in this connection, let $|B|$ be the cardinality of set B and, if $|B| > 0$ let b_j, $j = 1, \dots, |B|$, be the generic element of set B; thus, γ_{b_j}, $j = 1, \dots, |B|$, are the value of the horizontal axis at which the slope of $f(x)$ changes from a value greater than or equal to ν to a value less than ν.

Since it has been assumed $\mu_0 = 0$, then $a_j < b_j \; \forall \, j = 1, \dots, |B|$ and $b_j < a_{j+1} \; \forall \, j = 1, \dots, |A| - 1$. Moreover, $|A| - |B| \leq 1$ being $|A| = |B|$ if $\mu_M < \nu$ and $|A| = |B| + 1$ if $\mu_M \geq \nu$.

Lemma 1. Let $f(x+t)$ be a continuous nondecreasing piece-wise linear function of x, parameterized by t, as defined by definition 2, and let $g(x)$ be a noncontinuous function of x, as defined by definition 3.

In case $|B| \geq 1$, let Ω be the set of time instants $\{\omega_1, \ldots, \omega_j, \ldots, \omega_{|B|}\}$ in which any value ω_j, $j = 1, \ldots, |B|$, is obtained by executing algorithm 1. Each value ω_j, $j = 1, \ldots, |B|$, is either finite or nonfinite. Let T be the set of time instants $\{t_1^\star, \ldots, t_q^\star, \ldots, t_Q^\star\}$ which is obtained from Ω by removing all nonfinite values from it, that is

$$T = \Omega \setminus \{\omega_j \, : \, \omega_j = +\infty \, , \, j = 1, \ldots, |B|\} \tag{2}$$

Let Q be the cardinality of set T; it is obviously $1 \leq Q \leq |B|$. In case $|B| = 0$, it is $T = \emptyset$ and $Q = 0$.

Then, the function of t

$$x^\circ(t) = \arg \min_{\substack{x \\ x_1 \leq x \leq x_2}} \{f(x+t) + g(x)\} \tag{3}$$

is a nonincreasing, possibly noncontinuous, piece-wise linear function of t defined as

$$\text{if } Q = 0 : \quad x^\circ(t) = x_\mathrm{e}(t) \tag{4a}$$

$$\text{if } Q = 1 : \quad x^\circ(t) = \begin{cases} x_\mathrm{s}(t) & t < t_1^\star \\ x_\mathrm{e}(t) & t \geq t_1^\star \end{cases} \tag{4b}$$

$$\text{if } Q > 1 : \quad x^\circ(t) = \begin{cases} x_\mathrm{s}(t) & t < t_1^\star \\ x_q(t) & t_q^\star \leq t < t_{q+1}^\star, \quad q = 1, \ldots, Q-1 \\ x_\mathrm{e}(t) & t \geq t_Q^\star \end{cases} \tag{4c}$$

where $x_\mathrm{s}(t)$, $x_q(t)$, and $x_\mathrm{e}(t)$ are the following functions of t:

- $x_\mathrm{s}(t)$ is a continuous nonincreasing piece-wise linear functions of t defined as:

$$\text{if } t_1^\star > \gamma_{a_1} - x_1 \quad \rightarrow \quad x_\mathrm{s}(t) = \begin{cases} x_2 & t < \gamma_{a_1} - x_2 \\ -t + \gamma_{a_1} & \gamma_{a_1} - x_2 \leq t < \gamma_{a_1} - x_1 \\ x_1 & \gamma_{a_1} - x_1 \leq t < t_1^\star \end{cases} \tag{5a}$$

$$\text{if } t_1^\star \leq \gamma_{a_1} - x_1 \quad \rightarrow \quad x_\mathrm{s}(t) = \begin{cases} x_2 & t < \gamma_{a_1} - x_2 \\ -t + \gamma_{a_1} & \gamma_{a_1} - x_2 \leq t < t_1^\star \end{cases} \tag{5b}$$

- $x_q(t)$ is a continuous nonincreasing piece-wise linear functions of t defined as:

$$\begin{array}{l}\text{if } t_q^\star < \gamma_{a_{l(q)+1}} - x_2 \\ \text{and } t_{q+1}^\star > \gamma_{a_{l(q)+1}} - x_1\end{array} \rightarrow x_q(t) = \begin{cases} x_2 & t_q^\star \leq t < \gamma_{a_{l(q)+1}} - x_2 \\ -t + \gamma_{a_{l(q)+1}} & \gamma_{a_{l(q)+1}} - x_2 \leq t < \gamma_{a_{l(q)+1}} - x_1 \\ x_1 & \gamma_{a_{l(q)+1}} - x_1 \leq t < t_{q+1}^\star \end{cases} \tag{6a}$$

$$\begin{array}{l}\text{if } t_q^\star \geq \gamma_{a_{l(q)+1}} - x_2 \\ \text{and } t_{q+1}^\star > \gamma_{a_{l(q)+1}} - x_1\end{array} \rightarrow x_q(t) = \begin{cases} -t + \gamma_{a_{l(q)+1}} & t_q^\star \leq t < \gamma_{a_{l(q)+1}} - x_1 \\ x_1 & \gamma_{a_{l(q)+1}} - x_1 \leq t < t_{q+1}^\star \end{cases} \tag{6b}$$

$$\begin{array}{l}\text{if } t_q^\star < \gamma_{a_{l(q)+1}} - x_2 \\ \text{and } t_{q+1}^\star \leq \gamma_{a_{l(q)+1}} - x_1\end{array} \rightarrow x_q(t) = \begin{cases} x_2 & t_q^\star \leq t < \gamma_{a_{l(q)+1}} - x_2 \\ -t + \gamma_{a_{l(q)+1}} & \gamma_{a_{l(q)+1}} - x_2 \leq t < t_{q+1}^\star \end{cases} \tag{6c}$$

$$\begin{array}{l}\text{if } t_q^\star \geq \gamma_{a_{l(q)+1}} - x_2 \\ \text{and } t_{q+1}^\star \leq \gamma_{a_{l(q)+1}} - x_1\end{array} \rightarrow x_q(t) = -t + \gamma_{a_{l(q)+1}} \tag{6d}$$

- $x_\mathrm{e}(t)$ is a continuous nonincreasing piece-wise linear functions of t defined as:

$$\begin{array}{l}\text{if } l(Q) < |A| \\ \text{and } t_Q^\star < \gamma_{a_{l(Q)+1}} - x_2\end{array} \rightarrow x_\mathrm{e}(t) = \begin{cases} x_2 & t_Q^\star \leq t < \gamma_{a_{l(Q)+1}} - x_2 \\ -t + \gamma_{a_{l(Q)+1}} & \gamma_{a_{l(Q)+1}} - x_2 \leq t < \gamma_{a_{l(Q)+1}} - x_1 \\ x_1 & t \geq \gamma_{a_{l(Q)+1}} - x_1 \end{cases} \tag{7a}$$

$$\begin{array}{l}\text{if } l(Q) < |A| \\ \text{and } t_Q^\star \geq \gamma_{a_{l(Q)+1}} - x_2\end{array} \rightarrow x_\mathrm{e}(t) = \begin{cases} -t + \gamma_{a_{l(Q)+1}} & t_Q^\star \leq t < \gamma_{a_{l(Q)+1}} - x_1 \\ x_1 & t \geq \gamma_{a_{l(Q)+1}} - x_1 \end{cases} \tag{7b}$$

$$\text{if } l(Q) = |A| \quad \rightarrow \quad x_\mathrm{e}(t) = x_2 \tag{7c}$$

having assumed (for notational convenience) $t_Q^\star = -\infty$ when $Q = 0$.

In (6) and (7), $l(q)$, $q = 1, \dots, Q$, is a mapping function which provides the index $j \in \{1, \dots, |B|\}$ of the value $t_q^\star$ in the set Ω, that is, $l(q) = j \Leftrightarrow \omega_j = t_q^\star$. In this connection, it is always $l(Q) = |B|$ and, in case $|A| > |B|$, it turns out $l(Q) + 1 = |A|$. Moreover, it is assumed, for notational convenience, $l(0) = 0$.

Algorithm 1. Determination of the time instant ω_j, $j = 1, \dots, |B|$, at which, in case $\omega_j < +\infty$, the function $x^\circ(t)$ jumps in an upward direction.

SECTION A – INITIALIZATION

$\gamma_0 = -\infty$
$h \geq 0 \, : \, \gamma_h \leq \gamma_{b_j} - (x_2 - x_1) < \gamma_{h+1}$
$i = b_j$
$\gamma_{M+1} = +\infty$
$k \leq M \, : \, \gamma_k < \gamma_{b_j} + (x_2 - x_1) \leq \gamma_{k+1}$
if $j = |B|$ **and** $|A| = |B|$ **then**
    $a_{j+1} = M + 1$
end if
for $p = h$ **to** k **do**
    $\tilde{\mu}_p = \mu_p - \nu$
end for
$\tau = \gamma_{b_j} - (x_2 - x_1)$
$\theta = \gamma_{b_j}$
$d = \max\{0, \tilde{\mu}_h(\gamma_{h+1} - \tau)\}$
if $h < b_j - 1$ **then**
    for $p = h + 1$ **to** $b_j - 1$ **do**
        $d = \max\{0, d + \tilde{\mu}_p(\gamma_{p+1} - \gamma_p)\}$
    end for
end if
$\lambda = h$
$\xi = i$

SECTION B – FIRST LOOP

while $h < b_j$ **and** $i < a_{j+1}$ **do**
    $\psi = \min\{\gamma_{h+1} - \tau, \gamma_{i+1} - \theta\}$
    if $\gamma_{h+1} - \tau \leq \gamma_{i+1} - \theta$ **then**
        $\lambda = h + 1$
    end if
    if $\gamma_{h+1} - \tau \geq \gamma_{i+1} - \theta$ **then**
        $\xi = i + 1$
    end if
    $\delta = \max\{0, \tilde{\mu}_\lambda[\gamma_{\lambda+1} - (\tau + \psi)]\}$
    if $\lambda < b_j - 1$ **then**
        for $p = \lambda + 1$ **to** $b_j - 1$ **do**
            $\delta = \max\{0, \delta + \tilde{\mu}_p(\gamma_{p+1} - \gamma_p)\}$
        end for
    end if
    if $\xi = b_j$ **then**
        $\delta = \delta + \tilde{\mu}_\xi[(\theta + \psi) - \gamma_\xi]$
    else if $\xi = a_{j+1}$ **then**
        $\delta = \delta + \sum_{p=b_j}^{\xi-1} \tilde{\mu}_p(\gamma_{p+1} - \gamma_p)$
    else
        $\delta = \delta + \sum_{p=b_j}^{\xi-1} \tilde{\mu}_p(\gamma_{p+1} - \gamma_p) +$
            $+ \tilde{\mu}_\xi[(\theta + \psi) - \gamma_\xi]$
    end if
    if $\delta \leq 0$ **then**
        $a_0 = 0$
        $r \geq 1 \, : \, a_{r-1} \leq h < a_r$
        if $r \leq j$ **then**
            for $q = r$ **to** j **do**
                $\chi = \tilde{\mu}_h(\gamma_{h+1} - \tau)$
                if $h < a_q - 1$ **then**
                    $\chi = \chi + \sum_{p=h+1}^{a_q-1} \tilde{\mu}_p(\gamma_{p+1} - \gamma_p)$
                end if
                if $q = r$ **then**
                    $m = \chi$
                else
                    $m = \min\{m, \chi\}$
                end if
            end for
            if $m \leq 0$ **then**
                $\omega_j = \tau - x_1 - \frac{d}{\tilde{\mu}_i}$
            else if $-\frac{d-m}{\tilde{\mu}_i} \leq \frac{m}{\tilde{\mu}_h}$ **then**
                $\omega_j = \tau - x_1 + \frac{d}{\tilde{\mu}_h - \tilde{\mu}_i}$
                exit algorithm
            else
                $\omega_j = \tau - x_1 - \frac{d-m}{\tilde{\mu}_i}$
                exit algorithm
            end if
        else
            $\omega_j = \tau - x_1 + \frac{d}{\tilde{\mu}_h - \tilde{\mu}_i}$
            exit algorithm
        end if
    else
        $h = \lambda$
        $i = \xi$
        $\tau = \tau + \psi$
        $\theta = \theta + \psi$
        $d = \delta$
    end if
end while

SECTION C – SECOND LOOP

while $h < b_j$ **do**
    $\psi = \gamma_{h+1} - \tau$
    $\lambda = h + 1$
    if $\lambda < b_j$ **then**
        $\delta = \max\{0, \tilde{\mu}_\lambda[\gamma_{\lambda+1} - (\tau + \psi)]\}$
        if $\lambda < b_j - 1$ **then**
            for $p = \lambda + 1$ **to** $b_j - 1$ **do**
                $\delta = \max\{0, \delta + \tilde{\mu}_p(\gamma_{p+1} - \gamma_p)\}$
            end for
        end if
    else
        $\delta = 0$

end if
$\delta = \delta + \sum_{p=b_j}^{a_{j+1}-1} \tilde{\mu}_p(\gamma_{p+1} - \gamma_p)$
if $\delta \leq 0$ **then**
  $\tau = \tau + \frac{d}{\tilde{\mu}_h}$
  $\theta = \tau + (x_2 - x_1)$
  $k \leq M \, : \, \gamma_k < \theta \leq \gamma_{k+1}$
  $r = a_{j+1}$
  $\phi = 0$
  while $r \leq k$ **do**
    if $r < k$ **then**
      $\phi = \phi + \tilde{\mu}_r(\gamma_{r+1} - \gamma_r)$
    else
      $\phi = \phi + \tilde{\mu}_r(\theta - \gamma_r)$
    end if
    if $\phi < 0$ **then**
      $\omega_j = +\infty$
      exit algorithm
    else
      $r = r + 1$
    end if
  end while
  $\omega_j = \tau - x_1$
  exit algorithm
else
  $h = \lambda$
  $\tau = \tau + \psi$
  $d = \delta$
end if
end while

Proof. The function $x^{\circ}(t)$ can be obtained by analyzing the shape of the function $f(x+t) + g(x)$ in the interval $[x_1, x_2]$, with t moving from $-\infty$ to $+\infty$. The proof consists of seven parts:

1. in the first part, it is proven that, when $|B| = 0$, $x^{\circ}(t)$ has the structure provided by (4a), with $x_{\mathrm{e}}(t)$ provided by (7a) (if $l(Q) < |A|$) or (7c) (if $l(Q) = |A|$);

2. in the second part, it is proven that, when $\gamma_{b_j} - \gamma_{a_j} > (x_2 - x_1)$, $\forall\, j = 1, \ldots, |B|$, $|B| > 0$, and $\gamma_{a_{j+1}} - \gamma_{b_j} > (x_2 - x_1)$, $\forall\, j = 1, \ldots, |A|-1$, $|A| > 1$, $x^{\circ}(t)$ has the structure provided by (4b) or (4c), with $x_{\mathrm{s}}(t)$ provided by (5a), $x_q(t)$, $q = 1, \ldots, Q-1$, $Q > 1$, provided by (6a), and $x_{\mathrm{e}}(t)$ provided by (7a) (if $l(Q) < |A|$) or (7c) (if $l(Q) = |A|$);

3. in the third part, it is shown that the number of jump discontinuities in $x^{\circ}(t)$ may be less than $|B|$, that is, they are $Q \leq |B|$, and the conditions for which a jump discontinuity does not exist in connection with a specific abscissa $\gamma_{b_j} - t$, $j \in \{1, \ldots, |B|\}$ are provided;

4. in the fourth part, it is proven that, even if the assumptions considered in the second part do not hold for some $j \in \{1, \ldots, |B|\}$ or $j \in \{1, \ldots, |A| - 1\}$, it is sufficient that $t_1^{\star} > \gamma_{a_1} - x_1$, or $t_q^{\star} < \gamma_{a_{l(q)+1}} - x_2$ and $t_{q+1}^{\star} > \gamma_{a_{l(q)+1}} - x_1$, $q \in \{1, \ldots, Q-1\}$, $Q > 1$, or $t_Q^{\star} < \gamma_{a_{l(Q)+1}} - x_2$, to guarantee that $x_{\mathrm{s}}(t)$ has the structure provided by (5a), $x_q(t)$ has the structure provided by (6a), and $x_{\mathrm{e}}(t)$ has the structure provided by (7a) (if $l(Q) < |A|$), respectively;

5. in the fifth part, it is proven that, if $t_q^{\star} \geq \gamma_{a_{l(q)+1}} - x_2$, $q \in \{1, \ldots, Q-1\}$ or $t_Q^{\star} \geq \gamma_{a_{l(Q)+1}} - x_2$, then there is at $t = t_q^{\star}$ a discontinuity in $x^{\circ}(t)$ at which it jumps upwardly from x_1 to $-t + \gamma_{a_{l(q)+1}} \leq x_2$, $q \in \{1, \ldots, Q\}$, that is, $x_q(t)$ in (4c) has the structure of (6b) or (6d) or $x_{\mathrm{e}}(t)$ in (4b) and (4c) has the structure of (7b);

6. in the sixth part, it is proven that, if $t_1^{\star} \leq \gamma_{a_1} - x_1$ or $t_{q+1}^{\star} \leq \gamma_{a_{l(q)+1}} - x_1$, $q \in \{1, \ldots, Q-1\}$, then there is at $t = t_{q+1}^{\star}$ a discontinuity in $x^{\circ}(t)$ at which it jumps upwardly from $-t + \gamma_{a_{l(q)+1}} \geq x_1$ to x_2, $q \in \{1, \ldots, Q\}$, that is, $x_{\mathrm{s}}(t)$ in (4b) and (4c) has the structure of (5b) or $x_q(t)$ in (4c) has the structure of (6c) or (6d);

7. in the seventh and last part, algorithm 1, which allows determining time instants ω_j, $j = 1, \ldots, |B|$, is described.

First part

Consider the case $|B| = 0$, which implies $\Omega = \emptyset$ and then $T = \emptyset$ and $Q = 0$. Moreover, $l(0) = 0$. If $|A| = 0$ as well, all the slopes of $f(x+t)$ are less than ν; then, $f(x+t) + g(x)$ is a strictly decreasing function of x, $\forall\, t$. In this case, the minimum of the function $f(x+t) + g(x)$, with respect to $[x_1, x_2]$, is always obtained at x_2. Thus, in this case, $x^{\circ}(t)$ has the structure provided by (4a), with $x_{\mathrm{e}}(t)$ provided by (7c), being $l(Q) = |A|$.

If $|A| > 0$, it is definitely $|A| = 1 = l(Q) + 1$. In this case, the slopes of $f(x+t)$ are less than ν in the interval $(-\infty, \gamma_{a_{l(Q)+1}})$ and greater than or equal to ν in $[\gamma_{a_{l(Q)+1}}, +\infty)$. This case is very similar to that considered in

lemma 1 of [1]. When t is such that $x_2 < \gamma_{a_{l(Q)+1}} - t$ (that is, $t < \gamma_{a_{l(Q)+1}} - x_2$), the minimum with respect to x, $x_1 \leq x \leq x_2$, of $f(x+t) + g(x)$ is obtained at x_2. When t is such that $x_1 < \gamma_{a_{l(Q)+1}} - t \leq x_2$ (that is, $\gamma_{a_{l(Q)+1}} - x_2 \leq t < \gamma_{a_{l(Q)+1}} - x_1$), the function $f(x+t) + g(x)$ is strictly decreasing in $[x_1, \gamma_{a_{l(Q)+1}} - t]$ and nondecreasing in $[\gamma_{a_{l(Q)+1}} - t, x_2]$; then, it has a minimum, with respect to $[x_1, x_2]$, in $\gamma_{a_j} - t$; when t increases in the interval $[\gamma_{a_{l(Q)+1}} - x_2, \gamma_{a_{l(Q)+1}} - x_1)$, the minimum decreases (with unitary speed) from x_2 to x_1. Finally, when t is such that $x_1 \geq \gamma_{a_{l(Q)+1}} - t$ (that is, $t \geq \gamma_{a_{l(Q)+1}} - x_1$), the minimum is obtained at x_1. Thus, in this case, $x^\circ(t)$ has the structure provided by (4a), with $x_\mathrm{e}(t)$ provided by (7a), being $l(Q) < |A|$. Note that, since $t^\star_Q = -\infty$ when $Q = 0$, it is $t^\star_Q < \gamma_{a_{l(Q)+1}} - x_2$ for sure.

Second part

Consider the case $|A| > 1$ and $|B| > 0$, and assume $\gamma_{b_j} - \gamma_{a_j} > (x_2 - x_1)$, $\forall\, j = 1, \ldots, |B|$, and $\gamma_{a_{j+1}} - \gamma_{b_j} > (x_2 - x_1)$, $\forall j = 1, \ldots, |A| - 1$. Under such hypotheses, the function $x^\circ(t)$ is defined as follows.

1. When t is such that the slopes of $f(x+t)$ in the interval $[x_1, x_2]$ are less than ν, that is, $\forall\, t < \gamma_{a_1} - x_2$, $\forall\, t \in [\gamma_{b_j} - x_1, \gamma_{a_{j+1}} - x_2)$, $j = 1, \ldots, |A| - 1$, and $\forall\, t \geq \gamma_{b_{|B|}} - x_1$ if $|A| = |B|$, the minimum of the function $f(x+t) + g(x)$, with respect to $[x_1, x_2]$, is obtained at x_2, since $f(x+t) + g(x)$ is strictly decreasing in $[x_1, x_2]$.
2. When t is such that $\gamma_{a_j} - t \in [x_1, x_2]$, $j = 1, \ldots, |A|$, that is, $\forall\, t \in [\gamma_{a_j} - x_2, \gamma_{a_j} - x_1)$, $j = 1, \ldots, |A|$, the function $f(x+t) + g(x)$ is strictly decreasing in $[x_1, \gamma_{a_j} - t]$ and nondecreasing in $[\gamma_{a_j} - t, x_2]$; then, it has a minimum, with respect to $[x_1, x_2]$, in $\gamma_{a_j} - t$; when t increases in the interval $[\gamma_{a_j} - x_2, \gamma_{a_j} - x_1)$, the minimum decreases (with unitary speed) from x_2 to x_1.
3. When t is such that the slope of $f(x+t)$ in the interval $[x_1, x_2]$ is grater than or equal to ν, that is, $\forall\, t \in [\gamma_{a_j} - x_1, \gamma_{b_j} - x_2)$, $j = 1, \ldots, |B|$, and $\forall\, t \geq \gamma_{a_{|A|}} - x_1$ if $|A| > |B|$, the minimum of the function $f(x+t) + g(x)$, with respect to $[x_1, x_2]$, is obtained at x_1, since $f(x+t) + g(x)$ is nondecreasing in $[x_1, x_2]$.
4. When t is such that $\gamma_{b_j} - t \in [x_1, x_2]$, $j = 1, \ldots, |B|$, that is, $\forall\, t \in [\gamma_{b_j} - x_2, \gamma_{b_j} - x_1)$, $j = 1, \ldots, |B|$, the function $f(x+t) + g(x)$ is nondecreasing in $[x_1, \gamma_{b_j} - t]$ and strictly decreasing in $[\gamma_{b_j} - t, x_2]$; then, it has a maximum, with respect to $[x_1, x_2]$, in $x = \gamma_{b_j} - t$, and the minimum is obtained either at x_1 or x_2, depending on the values $f(x_1+t)+g(x_1)$ and $f(x_2+t)+g(x_2)$, $t \in [\gamma_{b_j} - x_2, \gamma_{b_j} - x_1)$ (the minimum is obtained at $x^\circ = x_1$ if $f(x_1+t)+g(x_1) < f(x_2+t)+g(x_2)$ and at $x^\circ = x_2$ otherwise). In this connection, note that:
   - when $t = \gamma_{b_j} - x_2$, it is certainly $f(x_1+t) + g(x_1) \leq f(x_2+t) + g(x_2)$;
   - when t increases in the interval $(\gamma_{b_j} - x_2, \gamma_{b_j} - x_1)$, the value of $f(x_1+t)+g(x_1)$ increases or remains constant and the value of $f(x_2+t) + g(x_2)$ decreases;
   - when $t = \gamma_{b_j} - x_1$, it is certainly $f(x_1+t) + g(x_1) > f(x_2+t) + g(x_2)$.

   This means that it certainly exists $\omega_j \in [\gamma_{b_j} - x_2, \gamma_{b_j} - x_1)$ such that $f(x_1+t)+g(x_1) \leq f(x_2+t)+g(x_2)$ $\forall\, t \in [\gamma_{b_j} - x_2, \omega_j)$, $f(x_1+\omega_j)+g(x_1) = f(x_2+\omega_j)+g(x_2)$, and $f(x_1+t)+g(x_1) > f(x_2+t)+g(x_2)$ $\forall\, t \in (\omega_j, \gamma_{b_j} - x_1)$; then, the minimum is obtained at x_1 $\forall\, t \in [\gamma_{b_j} - x_2, \omega_j)$, "jumps" from x_1 to x_2 at ω_j, and is obtained at x_2 $\forall\, t \in [\omega_j, \gamma_{b_j} - x_1)$.

Thus, according to the previous "rules", since $\mu_0 = 0$ the function $x^\circ(t)$ is x_2 at the beginning (rule 1), decreases with slope -1 in the interval $[\gamma_{a_1} - x_2, \gamma_{a_1} - x_1)$ (rule 2), is equal to x_1 from $\gamma_{a_1} - x_1$ to ω_1, at which it jumps to x_2 (rules 3 and 4); $x^\circ(t)$ remains equal to x_2 from ω_1 up to $\gamma_{a_2} - x_2$ (rules 4 and 1), then it descreases with slope -1 in the interval $[\gamma_{a_2} - x_2, \gamma_{a_2} - x_1)$ (rule 2), is equal to x_1 from $\gamma_{a_2} - x_1$ to ω_2, at which jumps to x_2 (rules 3 and 4), and so on. In its last part, the function $x^\circ(t)$ is x_1 if $|A| > |B|$ or x_2 if $|A| = |B|$, in accordance with rules 3 and 1.

Time instants $\omega_j \in [\gamma_{b_j} - x_2, \gamma_{b_j} - x_1)$, $j = 1, \ldots, |B|$, are those for which it results $f(x_1+\omega_j) + g(x_1) = f(x_2+\omega_j) + g(x_2)$. They can be determined through a simple procedure which analyzes the values within the interval $[x_1, x_2]$ of the piece-wise linear function $f(x+t)+g(x)$, during its leftward movement (when t increases from $t = \gamma_{b_j} - x_2$ up to $t = \gamma_{b_j} - x_1$). Consider figure 4 which illustrates an example of function $f(x+t)+g(x)$ when $t = \gamma_{b_j} - x_2$ (only the part which belongs to the interval $[x_2 - (x_2 - x_1), x_2 + (x_2 - x_1)] \equiv [x_1, 2x_2 - x_1]$ is reported, and note also that the vertical value is not meaningful in the search of the abscissa at which the minimum

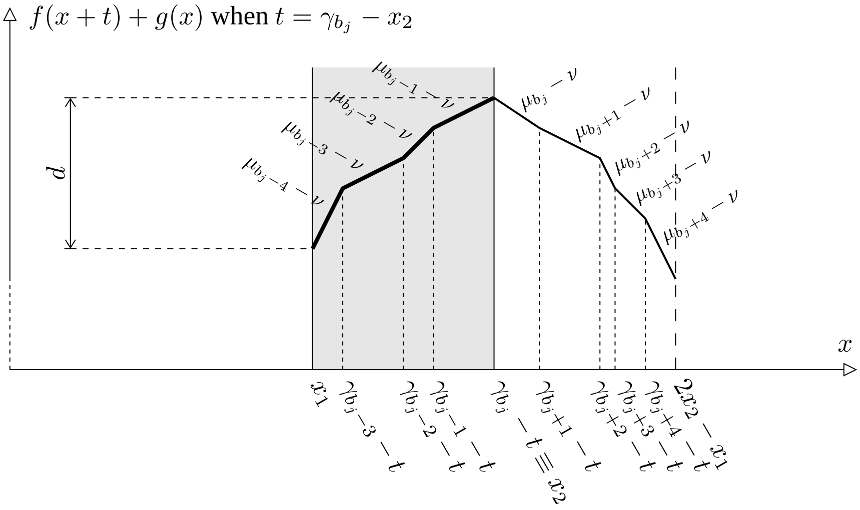


Figure 4: Example of function $f(x+t)+g(x)$, when $t=\gamma_{b_j}-x_2$.

is obtained). Let $d=f(x_2+t)+g(x_2)-f(x_1+t)+g(x_1)$. Without considering the upward movement of the function (which is not important for the determination of ω_j), when t increases the function moves leftward and d is reduced. As an example, in figure 5 the same function $f(x+t)+g(x)$ is illustrated when $t=\gamma_{b_j-3}-x_1$. It is evident that ω_j is the time instant at which d is null.

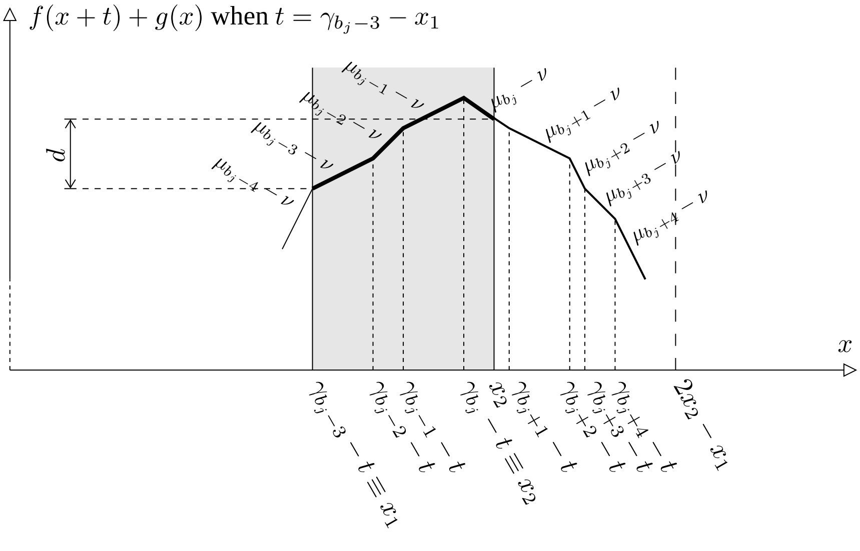


Figure 5: Example of function $f(x+t)+g(x)$, when $t=\gamma_{b_j-3}-x_1$.

On the basis of such considerations, with the considered assumptions, to compute ω_j it is possible to use the following algorithm (which is not formally described, but the reader can refer to the description of algorithm 1, which generalizes the following one).

SECTION A – INITIALIZATION

$h\in\{a_j,\dots,b_j-1\}$: $\gamma_h\le\gamma_{b_j}-(x_2-x_1)<\gamma_{h+1}$
$i=b_j$
if $j=|B|$ **and** $|A|=|B|$ **then**
    $a_{j+1}=M+1$
    $\gamma_{M+1}=+\infty$
end if
$k\in\{b_j,\dots,a_{j+1}-1\}$: $\gamma_k<\gamma_{b_j}+(x_2-x_1)\le\gamma_{k+1}$
for $p=h$ **to** k **do**
    $\tilde{\mu}_p=\mu_p-\nu$
end for
$\tau=\gamma_{b_j}-(x_2-x_1)$
$\theta=\gamma_{b_j}$
$d=\tilde{\mu}_h(\gamma_{h+1}-\tau)$
if $h<b_j-1$ **then**
    $d=d+\sum_{p=h+1}^{b_j-1}\tilde{\mu}_p(\gamma_{p+1}-\gamma_p)$

end if
$\lambda = h$
$\xi = i$

SECTION B – LOOP

while $h < b_j$ **and** $i < k+1$ **do**
  $\psi = \min\{\gamma_{h+1} - \tau, \gamma_{i+1} - \theta\}$
  if $\gamma_{h+1} - \tau \leq \gamma_{i+1} - \theta$ **then**
    $\lambda = h + 1$
  end if
  if $\gamma_{h+1} - \tau \geq \gamma_{i+1} - \theta$ **then**
    $\xi = i + 1$
  end if
  $\delta = \tilde{\mu}_\lambda[\gamma_{\lambda+1} - (\tau + \psi)]$
  if $\lambda < \xi - 1$ **then**
    $\delta = \delta + \sum_{p=\lambda+1}^{\xi-1} \tilde{\mu}_p(\gamma_{p+1} - \gamma_p)$
  end if
  $\delta = \delta + \tilde{\mu}_\xi[(\theta + \psi) - \gamma_\xi]$
  if $\delta \leq 0$ **then**
    $\omega_j = \tau - x_1 + \frac{d}{\tilde{\mu}_h - \tilde{\mu}_i}$
    exit algorithm
  else
    $h = \lambda$
    $i = \xi$
    $\tau = \tau + \psi$
    $\theta = \theta + \psi$
    $d = \delta$
  end if
end while

This algorithm provides, for any $j = 1, \ldots, |B|$, the time instant ω_j at which a jump discontinuity in $x^\circ(t)$ occurs. Since $\omega_j < +\infty\ \forall\, j = 1, \ldots, |B|$, then $T = \Omega$, being $\Omega = \{\omega_1, \ldots, \omega_j, \ldots, \omega_{|B|}\}$. Moreover, $Q = |B|$, $t_q^\star = \omega_q$, and $l(q) = q$, $\forall\, q = 1, \ldots, Q$. Then, it is possible to write $t_q^\star \in [\gamma_{b_{l(q)}} - x_2, \gamma_{b_{l(q)}} - x_1)$, $\gamma_{b_{l(q)}} - \gamma_{a_{l(q)}} > (x_2 - x_1)$, $\forall\, q = 1, \ldots, Q$, and $\gamma_{a_{l(q)+1}} - \gamma_{b_{l(q)}} > (x_2 - x_1)$, $\forall\, q = 1, \ldots, Q$ (if $l(Q) < |A|$) or $\forall\, q = 1, \ldots, Q - 1$ (if $l(Q) = |A|$), which imply $t_q^\star > \gamma_{a_{l(q)}} - x_1\ \forall\, q = 1, \ldots, Q$ and $t_q^\star < \gamma_{a_{l(q)+1}} - x_2\ \forall\, q = 1, \ldots, Q$ (if $l(Q) < |A|$) or $\forall\, q = 1, \ldots, Q - 1$ (if $l(Q) = |A|$).

Then, $x^\circ(t)$ has the structure provided by (4b) or (4c), with $x_{\mathrm{s}}(t)$ provided by (5a), $x_q(t)$, $q = 1, \ldots, Q - 1$, $Q > 1$, provided by (6a), and $x_{\mathrm{e}}(t)$ provided by (7a) (if $l(Q) < |A|$) or (7c) (if $l(Q) = |A|$).

Third part

It has been shown in the second part of the proof that, under the assumptions $\gamma_{b_j} - \gamma_{a_j} > (x_2 - x_1)$, $\forall\, j = 1, \ldots, |B|$, $|B| > 0$, and $\gamma_{a_{j+1}} - \gamma_{b_j} > (x_2 - x_1)$, $\forall j = 1, \ldots, |A| - 1$, $|A| > 1$, there exists, for each value b_j, $j = 1, \ldots, |B|$, a finite value $\omega_j \in [\gamma_{b_j} - x_2, \gamma_{b_j} - x_1)$ at which $x^\circ(t)$ jumps to x_2. In other words, there are $|B|$ points of discontinuity in the function $x^\circ(t)$. In presence of a narrower intervals, this is not necessarily true.

As a matter of fact, in connection with two consecutive time intervals $[\gamma_{b_j} - t, \gamma_{a_{j+1}} - t)$ and $[\gamma_{a_{j+1}} - t, \gamma_{b_{j+1}} - t)$ which are such that $\gamma_{b_{j+1}} - \gamma_{b_j} < (x_2 - x_1)$, when, for any $t \in [\gamma_{a_{j+1}} - x_2, \min\{\gamma_{b_j} - x_1, \gamma_{a_{j+2}} - x_2\})$, at least one of the two conditions $f(x_1 + t) + g(x_1) < f(\gamma_{a_{j+1}}) + g(\gamma_{a_{j+1}} - t)$ and $f(x_2 + t) + g(x_2) < f(\gamma_{a_{j+1}}) + g(\gamma_{a_{j+1}} - t)$ is satisfied, then the local minimum at $\gamma_{a_{j+1}} - t$ is never the absolute minimum in the interval $[x_1, x_2]$. Then, in this case, the presence of the abscissa γ_{b_j}, at which the slope of $f(x + t)$ changes from a value greater than or equal to ν to a value less than ν, does not cause the function $x^\circ(t)$ to jump in an upward direction.

To show this, consider the example of function $f(x + t) + g(x)$ illustrated in figure 6(a), when $t = \gamma_{b_j} - x_2$, in which it is $\gamma_{b_{j+1}} - \gamma_{b_j} < (x_2 - x_1)$. When $t = \gamma_{b_j} - x_2$, the minimum with respect to $[x_1, x_2]$ is obtained at x_1. If $f(x_1 + t) + g(x_1) < f(\gamma_{a_{j+1}}) + g(\gamma_{a_{j+1}} - t)$ when $t = \gamma_{a_{j+1}} - x_2$ (see figure 6(b)), that is, if $f(\gamma_{a_{j+1}} - x_2 + x_1) + g(x_1) < f(\gamma_{a_{j+1}})$, then the minimum certainly remains at x_1 when t increases in the interval $[\gamma_{b_j} - x_2, \gamma_{a_{j+1}} - x_2)$, since $f(x + t) + g(x)$ is strictly decreasing in $[\gamma_{b_j} - t, \gamma_{a_{j+1}} - t)$.

When t increases in the interval $[\gamma_{a_{j+1}} - x_2, \gamma_{b_{j+1}} - x_2)$, if $f(x_1 + t) + g(x_1) < f(\gamma_{a_{j+1}}) + g(\gamma_{a_{j+1}} - t)$ for all t in such an interval, then the minimum is once more at x_1, since $f(x + t) + g(x)$ is nondecreasing in $[\gamma_{a_{j+1}} - t, \gamma_{b_{j+1}} - t)$. When t increases in the interval $[\gamma_{b_{j+1}} - x_2, \gamma_{a_{j+2}} - x_2)$, if at least one of the two conditions $f(x_1 + t) + g(x_1) < f(\gamma_{a_{j+1}}) + g(\gamma_{a_{j+1}} - t)$ and $f(x_2 + t) + g(x_2) < f(\gamma_{a_{j+1}}) + g(\gamma_{a_{j+1}} - t)$ is satisfied, then the minimum is at x_1 (if $f(x_1 + t) + g(x_1) < f(x_2 + t) + g(x_2)$) or at x_2 or $\gamma_{a_{j+2}} - t$ (if $f(x_1 + t) + g(x_1) \geq f(x_2 + t) + g(x_2)$). More specifically, the minimum jumps from x_1 to x_2 (or to $\gamma_{a_{j+2}} - t$) when t is such that $f(x_1 + t) + g(x_1) = f(x_2 + t) + g(x_2)$ (see figure 6(c)); however, such a jump in an upward direction has to be associated with abscissa $\gamma_{b_{j+1}}$ and not with γ_{b_j}.

Note that, assumption $\gamma_{b_{j+1}} - \gamma_{b_j} < (x_2 - x_1)$ is a necessary condition, because in case $\gamma_{b_{j+1}} - \gamma_{b_j} \geq (x_2 - x_1)$ there is definitely a time instant t at which the local minimum at $\gamma_{a_{j+1}} - t$ is the absolute minimum in the interval $[x_1, x_2]$.

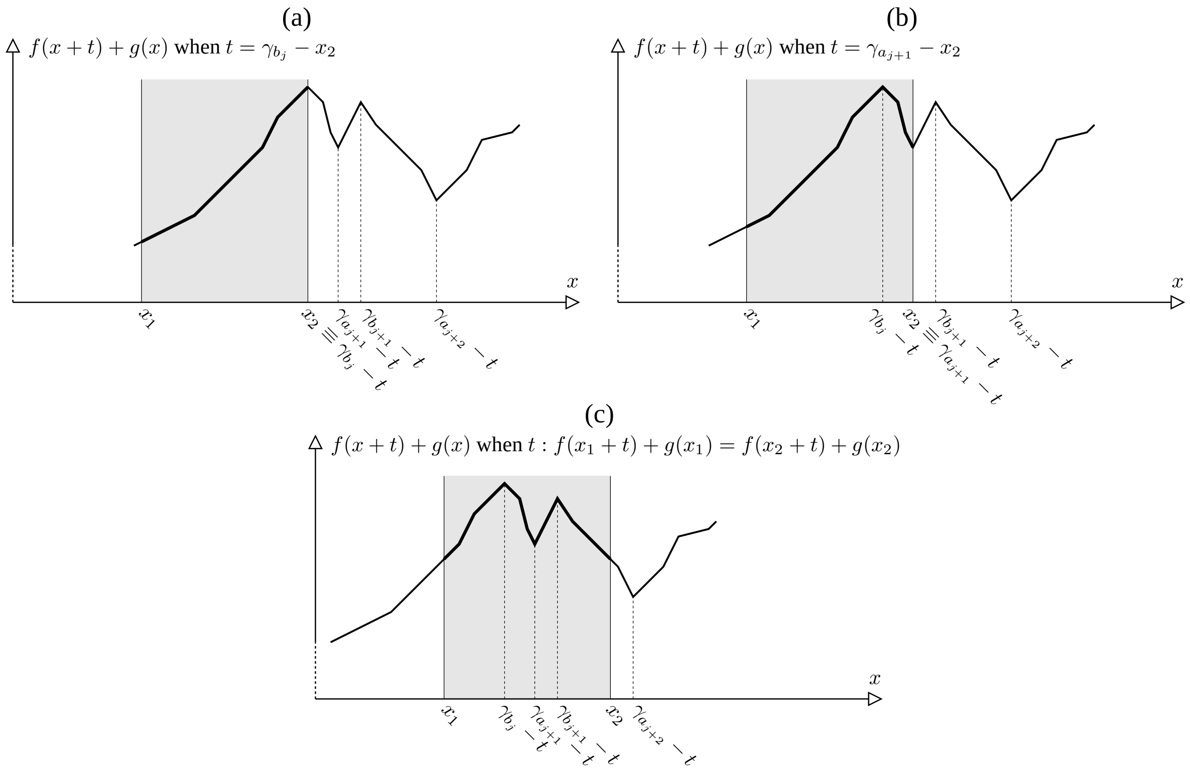


Figure 6: Example of function $f(x+t)+g(x)$, (a) when $t=\gamma_{b_j}-x_2$, (b) when $t=\gamma_{a_{j+1}}-x_2$, and (c) when $t: f(x_1+t)+g(x_1)=f(x_2+t)+g(x_2)$.

Algorithm 1 determines in the section C ("second loop") if the local minimum at $\gamma_{a_{j+1}}-t$ is the absolute minimum in the interval $[x_1, x_2]$. Such a part of the algorithm (rows 79÷ 92 and 114÷ 119) moves the function $f(x+t)+g(x)$ in a leftward direction (by increasing the time variable τ) until that the value of the function at the local minimum $\gamma_{a_{j+1}}-t$ is lower than or equal to the value of the function at x_1 or, equivalently, until that $f(x_1+t)+g(x_1) \geq f(\gamma_{a_{j+1}})+g(\gamma_{a_{j+1}}-t)$. When this happens, it results $\delta \leq 0$. At that point, the algorithm determines (at rows 93÷ 113) if there is a value of the function $f(x+t)+g(x)$ in $(\gamma_{a_{j+1}}-t, x_2] \subset [x_1, x_2]$ which is lower than the value of the function at the local minimum $\gamma_{a_{j+1}}-t$ or, equivalently, if it exists t such that $f(x_2+t)+g(x_2) < f(\gamma_{a_{j+1}})+g(\gamma_{a_{j+1}}-t)$. In the algorithm, such a lower value exists when $\phi < 0$; in this case, ω_j is set to the nonfinite value $+\infty$ and the algorithm ends since there is no more the possibility that the local minimum becomes an absolute minimum (since $f(x+t)+g(x)$ is strictly decreasing in $[\gamma_{b_{j+1}}-t, \gamma_{a_{j+2}}-t)$).

In conclusion, it has been shown in this part of the proof that some of the abscissae $\gamma_{b_j}-t$, $j \in \{1,\ldots,|B|\}$, may not cause a jump discontinuity in $x^{\circ}(t)$. Then, $x^{\circ}(t)$ has a number of discontinuities (at which it jumps in an upward direction) equal to $Q \leq |B|$. In accordance with the notation adopted in algorithm 1, values $\omega_j < +\infty$, $j \in \{1,\ldots,|B|\}$, are those actually corresponding to jump discontinuities. These Q values are denoted as $t_q^{\star}$, $q = 1,\ldots,Q$. The link between values ω_j and $t_q^{\star}$ is represented by the mapping function $l(q)=j$, $j \in \{1,\ldots,|B|\}$, $q=1,\ldots,Q$ (that is, $l(q)=j \Leftrightarrow \omega_j = t_q^{\star}$).

From now on, only the Q abscissae $\gamma_{b_{l(q)}}-t$, $q=1,\ldots,Q$, and the $Q+1$ abscissae γ_{a_1} and $\gamma_{a_{l(q)+1}}-t$, $q=1,\ldots,Q$, will be taken into account, to prove that, when $Q \geq 1$, $x^{\circ}(t)$ has the structure provided by (4b) or (4c), with $x_{\mathrm{s}}(t)$ provided by one of the (5), $x_q(t)$, $q=1,\ldots,Q-1$, $Q>1$, provided by one of the (6), and $x_{\mathrm{e}}(t)$ provided by one of the (7).

Fourth part

Consider the case $Q \geq 1$, and assume that $\gamma_{a_{l(q)+1}}-\gamma_{b_{l(q)}} < (x_2-x_1)$ and $\gamma_{b_{l(q)+1}}-\gamma_{a_{l(q)+1}} < (x_2-x_1)$ for some $q \in \{1,\ldots,Q\}$. An example of such a case is illustrated in figure 7. Note that, in accordance with the considerations made in the third part of the proof, regarding the time instants which actually produce a jump discontinuity in $x^{\circ}(t)$, such assumptions imply $l(q)+1 = l(q+1)$ and $l(q)+2 = l(q+1)+1 = l(q+2)$.

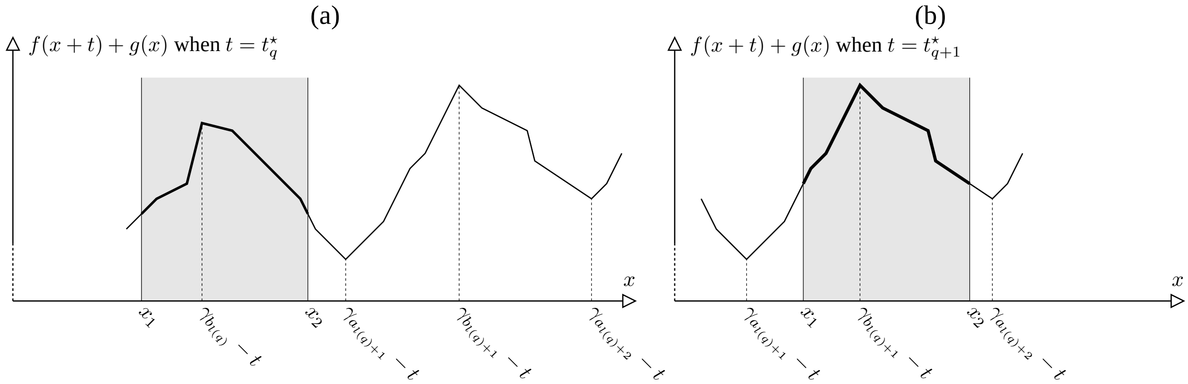


Figure 7: Example of function $f(x+t)+g(x)$, (a) when $t=t^\star_q$ and (b) when $t=t^\star_{q+1}$.

The condition $t^\star_q < \gamma_{a_{l(q)+1}} - x_2$ means that a discontinuity occurs at $t = t^\star_q$, at which $x^\circ(t)$ jumps to x_2, from either x_1 or $-t^\star_q+\gamma_{a_{l(q)}}$ (depending if $t^\star_q > \gamma_{a_{l(q)}} - x_1$ or $t^\star_q \le \gamma_{a_{l(q)}} - x_1$, respectively). In figure 7, $f(x+t)+g(x)$ when $t = t^\star_q$ is illustrated. In accordance with the rules discussed in the second part of the proof, when t increases from $t^\star_q$ to $\gamma_{a_{l(q)+1}} - x_2$, the minimum remains at x_2. Moreover, when t increases from $\gamma_{a_{l(q)+1}} - x_2$ on, the minimum decreases with unitary speed from x_2 towards x_1.

The condition $t^\star_{q+1} > \gamma_{a_{l(q)+1}} - x_1$ means that the minimum, which is decreasing, reaches x_1 when $t = \gamma_{a_{l(q)+1}} - x_1$, and remains at x_1 in the interval $[\gamma_{a_{l(q)+1}} - x_1, t^\star_{q+1})$. At $t = t^\star_{q+1}$ a discontinuity occurs, at which $x^\circ(t)$ jumps from x_1, to either x_2 or $-t^\star_{q+1} + \gamma_{a_{l(q)+2}}$ (depending if $t^\star_{q+1} < \gamma_{a_{l(q)+2}} - x_2$ or $t^\star_{q+1} \ge \gamma_{a_{l(q)+2}} - x_2$, respectively).

Then, in case $t^\star_q < \gamma_{a_{l(q)+1}} - x_2$ and $t^\star_{q+1} > \gamma_{a_{l(q)+1}} - x_1$, the function $x^\circ(t)$ between time instants $t^\star_q$ and $t^\star_{q+1}$ has the structure provided by (6a).

When $q = 0$, the function $f(x+t)+g(x)$ is strictly decreasing in $(-\infty, \gamma_{a_1} - t)$; then, the minimum is obtained at x_2 for all $t < \gamma_{a_1} - x_2$. When t increases from $\gamma_{a_1} - x_2$ on, the minimum decreases with unitary speed from x_2 towards x_1. The condition $t^\star_1 > \gamma_{a_1} - x_1$ means that the minimum, which is decreasing, reaches x_1 when $t = \gamma_{a_1} - x_1$, and remains at x_1 in the interval $[\gamma_{a_1} - x_1, t^\star_1)$. At $t = t^\star_1$ a discontinuity occurs, at which $x^\circ(t)$ jumps from x_1, to either x_2 or $-t^\star_1 + \gamma_{a_2}$ (depending if $t^\star_1 < \gamma_{a_2} - x_2$ or $t^\star_1 \ge \gamma_{a_2} - x_2$, respectively). Then, in case $t^\star_1 > \gamma_{a_1} - x_1$, the function $x^\circ(t)$ before time instant $t^\star_1$ has the structure provided by (5a).

When $q = Q$, if $t^\star_Q < \gamma_{a_{l(Q)+1}} - x_2$, then a discontinuity occurs at $t = t^\star_Q$, at which $x^\circ(t)$ jumps to x_2, from either x_1 or $-t^\star_Q + \gamma_{a_{l(Q)}}$ (depending if $t^\star_Q > \gamma_{a_{l(Q)}} - x_1$ or $t^\star_Q \le \gamma_{a_{l(Q)}} - x_1$, respectively). In accordance with the previous considerations, the minimum remains at x_2 in the interval $[t^\star_Q, \gamma_{a_{l(Q)+1}} - x_2)$, decreases with unitary speed in the interval $[\gamma_{a_{l(Q)+1}} - x_2, \gamma_{a_{l(Q)+1}} - x_1)$, and remains at x_1 from $t = \gamma_{a_{l(Q)+1}} - x_1$ on, since $f(x+t)+g(x)$ is nondecreasing in $[\gamma_{a_{l(Q)+1}} - x_1, +\infty)$. Then, in case $t^\star_Q < \gamma_{a_{l(Q)+1}} - x_2$, the function $x^\circ(t)$ after time instant $t^\star_Q$ has the structure provided by (7a).

<u>Fifth part</u>

Consider the case $Q \ge 1$, and assume that $\gamma_{a_{l(q)+1}} - \gamma_{b_{l(q)}} < (x_2 - x_1)$ and $\gamma_{a_{l(q)+1)}} - \gamma_{a_{l(q)}} > (x_2 - x_1)$ for some $q \in \{1, \ldots, Q-1\}$. If $t^\star_q \ge \gamma_{a_{l(q)+1}} - x_2$, then $f(x_1+t)+g(x_1) \le f(x_2+t)+g(x_2)$ when $t = \gamma_{a_{l(q)+1}} - x_2$, that is, $f(\gamma_{a_{l(q)+1}} - x_2 + x_1) + g(x_1) \le f(\gamma_{a_{l(q)+1}})$, as in the case illustrated in figure 8(a). When t increases from $\gamma_{a_{l(q)+1}} - x_2$ on, the local minimum at $\gamma_{a_{l(q)+1}} - t$ decreases with unitary speed from x_2 towards x_1. Thus, $t^\star_q$, which corresponds to the finite value $\omega_{l(q)}$, is the time instant at which $f(x_1+t)+g(x_1) = f(\gamma_{a_{l(q)+1}}) + g(\gamma_{a_{l(q)+1}} - t)$, as it is illustrated in figure 8(b). At $t^\star_q$, the minimum within $[x_1, x_2]$ jumps from x_1 to $-t^\star_q + \gamma_{a_{l(q)+1}}$. Then, $x_q(t)$ in (4c) has the structure of (6b) or (6d) (depending on the value $t^\star_{q+1}$, as discussed in the following part of the proof).

Consider now the same case in which $\gamma_{a_{l(q)+1}} - \gamma_{b_{l(q)}} < (x_2 - x_1)$, for some $q \in \{1, \ldots, Q-1\}$, but without any assumption about the interval $[\gamma_{a_{l(q)}}, \gamma_{a_{l(q)+1}})$. In this case, when $t = \gamma_{a_{l(q)+1}} - x_2$, one or more local minima are present in the interval $[x_1, x_2]$, as in the cases illustrated in figure 9(a). In accordance with the considerations made in the third part of the proof, regarding the time instants which actually produce a jump discontinuity in $x^\circ(t)$, if $t^\star_q \ge \gamma_{a_{l(q)+1}} - x_2$, then, when $t = \gamma_{a_{l(q)+1}} - x_2$, the global minimum in $[x_1, x_2]$ is at abscissa $\gamma_{a_{l(q-1)+1}} - t$ (see again figure 9(a)). As before, when t increases from $\gamma_{a_{l(q)+1}} - x_2$ on, the local minimum at $\gamma_{a_{l(q)+1}} - t$

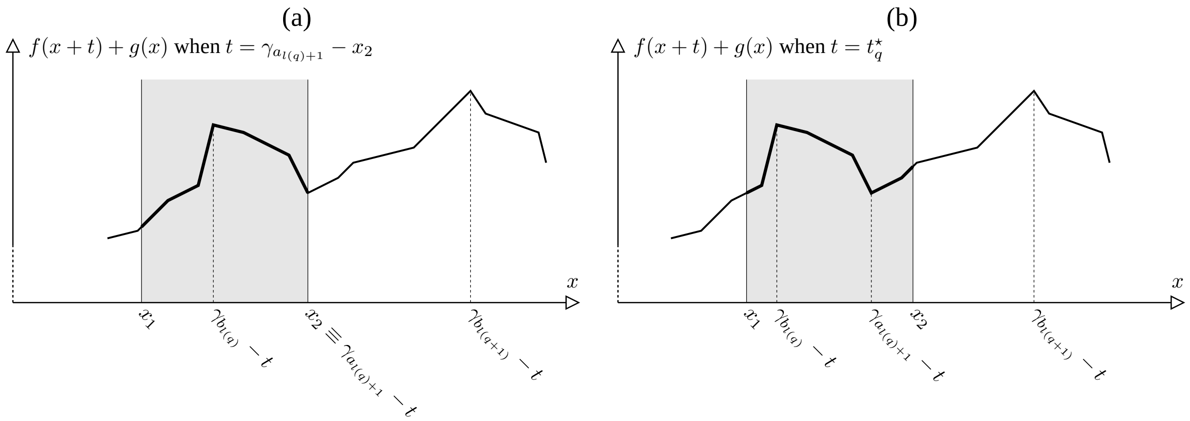


Figure 8: Example of function $f(x+t)+g(x)$, (a) when $t=\gamma_{a_{l(q)+1}}-x_2$ and (b) when $t=t^\star_q$.

decreases with unitary speed from x_2 towards x_1, and $t^\star_q$ is the time instant at which $f(x_1+t)+g(x_1)=f(\gamma_{a_{l(q)+1}})+g(\gamma_{a_{l(q)+1}}-t)$, as it is illustrated in figure 8(b). At $t^\star_q$, the minimum within $[x_1,x_2]$ jumps from x_1 to $-t^\star_q+\gamma_{a_{l(q)+1}}$. Then, $x_q(t)$ in (4c) has the structure of (6b) or (6d) (depending on the value $t^\star_{q+1}$).

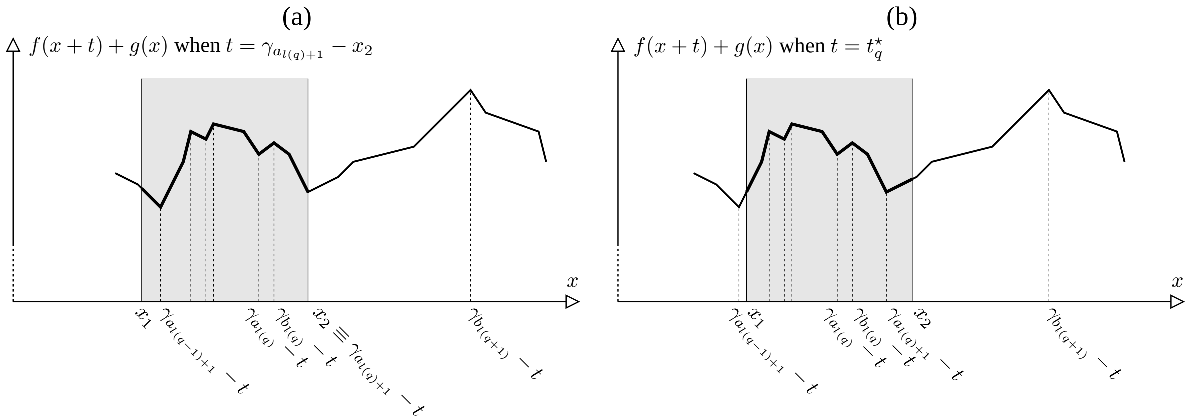


Figure 9: Example of function $f(x+t)+g(x)$, (a) when $t=\gamma_{a_{l(q)+1}}-x_2$ and (b) when $t=t^\star_q$.

The same considerations can be made when $q=Q$, in the case $l(Q)<|A|$. If $t^\star_Q\geq\gamma_{a_{l(Q)+1}}-x_2$, at $t^\star_Q$ the minimum within $[x_1,x_2]$ jumps from x_1 to $-t^\star_Q+\gamma_{a_{l(Q)+1}}$, and then $x_{\mathrm{e}}(t)$ in (4b) or (4c) has the structure of (7b).

Sixth part

Consider the case $Q\geq 1$, and assume that $\gamma_{b_{l(q+1)}}-\gamma_{a_{l(q)+1}}<(x_2-x_1)$ and $\gamma_{a_{l(q+1)+1}}-\gamma_{a_{l(q+1)}}>(x_2-x_1)$ for some $q\in\{1,\ldots,Q-1\}$. When $t=\gamma_{b_{l(q+1)}}-x_2$, the minimum within $[x_1,x_2]$ (which is decreasing with unitary speed since t was equal to $\gamma_{a_{l(q)+1}}-x_2$) is at $\gamma_{a_{l(q)+1}}-t$, as in the case illustrated in figure 10(a). If $t^\star_{q+1}\leq\gamma_{a_{l(q)+1}}-x_1$, then the minimum jumps to x_2 before than (or exactly when) it reaches x_1, that is, the minimum jumps from $\gamma_{a_{l(q)+1}}-t\geq x_1$ to x_2. $t^\star_{q+1}$ is the time instant at which $f(\gamma_{a_{l(q)+1}})+g(\gamma_{a_{l(q)+1}}-t)=f(x_2+t)+g(x_2)$, as it is illustrated in figure 8(b). Then, $x_q(t)$ in (4c) has the structure of (6c) or (6d) (depending on the value $t^\star_q$, as discussed in the previous part of the proof).

Consider now the same case in which $\gamma_{b_{l(q+1)}}-\gamma_{a_{l(q)+1}}<(x_2-x_1)$, for some $q\in\{1,\ldots,Q-1\}$, but without any assumption about the interval $[\gamma_{a_{l(q+1)}},\gamma_{a_{l(q+1)+1}})$. In this case, when $t=\gamma_{b_{l(q+1)}}-x_2$, one or more local minima are present in the interval $[x_1,x_2]$, as in the cases illustrated in figure 11(a). In accordance with the considerations made in the third part of the proof, regarding the time instants which actually produce a jump discontinuity in $x^\circ(t)$, the global minimum in $[x_1,x_2]$ is at abscissa $\gamma_{a_{l(q)+1}}-t$, as before. Then, also in this case, if $t^\star_{q+1}\leq\gamma_{a_{l(q)+1}}-x_1$, then the minimum jumps, at $t^\star_{q+1}$, from $\gamma_{a_{l(q)+1}}-t\geq x_1$ to x_2. In conclusion, $x_q(t)$ in (4c) has the structure of (6c) or (6d) (depending on the value $t^\star_q$).

The same considerations can be made in connection with time instant $t^\star_1$, when $\gamma_{b_{l(1)}}-\gamma_{a_1}<(x_2-x_1)$. In this case, if $t^\star_1\leq\gamma_{a_1}-x_1$, then the minimum jumps from $\gamma_{a_1}-t\geq x_1$ to x_2, and then $x_{\mathrm{s}}(t)$ in (4b) or (4c) has the

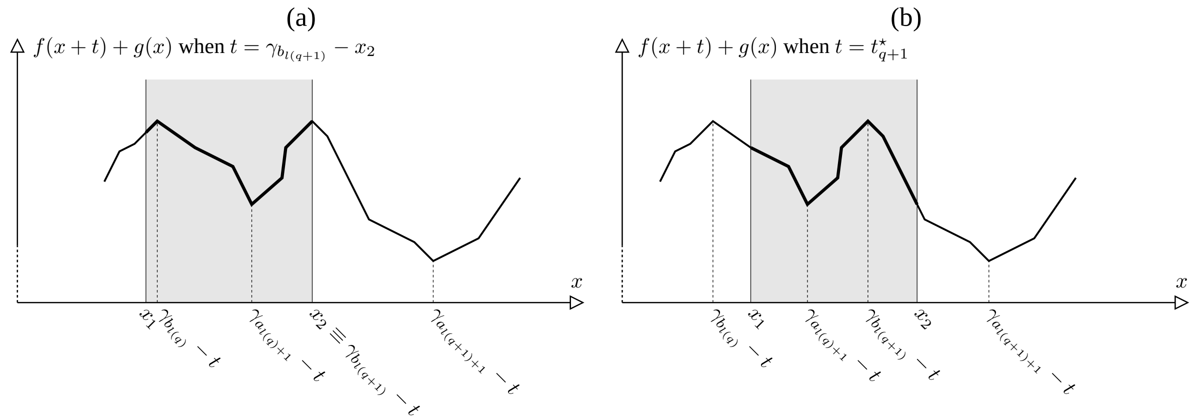


Figure 10: Example of function $f(x+t)+g(x)$, (a) when $t=\gamma_{b_{l(q+1)}}-x_2$ and (b) when $t=t^\star_{q+1}$.

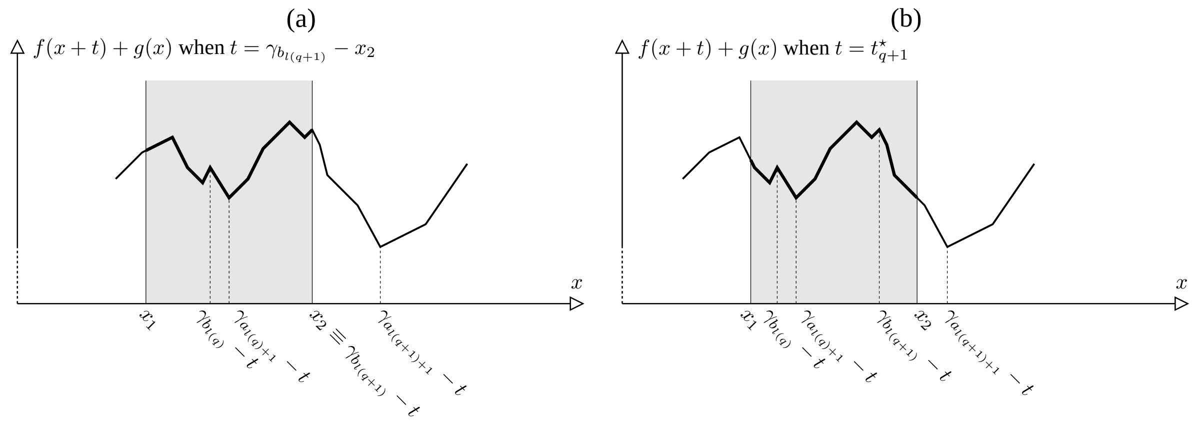


Figure 11: Example of function $f(x+t)+g(x)$, (a) when $t=\gamma_{b_{l(q+1)}}-x_2$ and (b) when $t=t^\star_{q+1}$.

structure of (5b).

Seventh part

The algorithm which computes the value ω_j, in correspondence with abscissa γ_{b_j}, considers the function $f(x)+g(x)$ and the "window" $[\gamma_{b_j}-(x_2-x_1),\gamma_{b_j}]$, which is moved rightward to find the instant at which the minimum of the function within the window "jumps" from the left bound to the right bound, as discussed in the previous parts of the proof. Note that, considering the function $f(x)+g(x)$ and the window $[\gamma_{b_j}-(x_2-x_1),\gamma_{b_j}]$ is equivalent to consider the function $f(x+t)+g(x)$ with $t=\gamma_{b_j}-x_2$ and the window $[x_1,x_2]$.

Basically, to determine the time instant at which the minimum within the window jumps in an upward direction, if it exists (as discussed in the third part of the proof), the algorithm moves the window rightward until the difference d between the value of $f(x)+g(x)$ at the right bound θ of the window (or at the local minimum which is the nearest to the right bound) and its value at left bound τ of the same window (or at the current global minimum within $[\tau,\theta]$) becomes null. Since $f(x)+g(x)$ is a piece-wise linear function, the window is repeatedly moved of intervals whose lengths ψ correspond to the lengths on the abscissae axis of the segments of the function. At each step, the new difference δ is computed and, if δ turns out to be null or negative, then the minimum has jumped to the right bound; this also means that the time instant ω_j is within the last rightward movement, that is, $\omega_j\in[\tau-x_1,\tau-x_1+\psi)$.

In the "Section A – Initialization" part of the algorithm, the segments of the piece-wise linear function $f(x)+g(x)$ which are included in the interval $[\gamma_{b_j}-(x_2-x_1),\gamma_{b_j}]$, and those of the interval $[\gamma_{b_j},\gamma_{b_j}+(x_2-x_1)]$ that could "enter" the window when it moves rightward, are determined (rows 1$\div$5); the slopes of $f(x)+g(x)$ are computed for any of those segments (rows 6$\div$11); the initial values of the left and right bounds τ and θ are set (rows 12$\div$13), and the initial value of d is calculated (rows 14$\div$19). Note that, the $\min$ operator in the determination of d is necessary to compute d when the minimum within $[\gamma_{b_j}-(x_2-x_1),\gamma_{b_j}]$ (that is, at the beginning) is not obtained

at the left bound but is obtained at an abscissa greater than $\gamma_{b_j} - (x_2 - x_1)$ (as for example, in the cases illustrated in figures 10(a) and 11(a)).

In the "Section B – First loop" of the algorithm, the while loop allows moving, segment-by-segment, the window $[\tau, \theta]$ leftward. At each step of the while loop, the length ψ of the next rightward movement is determined (row 23) and the new difference δ is computed (rows 30÷42). If $\delta \leq 0$, then ω_j can be determined (through one of the equation at rows 59, 61, 64 and 68); otherwise all values and indexes are updated (rows 72÷76) and another step of the loop is executed. It is worth noting that several equations to compute ω_j must be provided because of the possible presence of local minima within the moving window $[\tau, \theta]$; in this connection, values χ (rows 48÷51) are the relative value at the local minima (relative with respect to the value at the left bound of the window), and m (rows 52÷56) is the relative value at the global minimum; note also that all local minima, if present, are before $\gamma_{b_j} \in [\tau, \theta]$.

In the first loop, the window is moved until its right bound reaches abscissa $\gamma_{a_{j+1}}$. This means that, if ω_j is determined within the first loop, then the new minimum is definitely obtained at x_2, since $f(x) + g(x)$ is strictly decreasing in $[\gamma_{b_j}, \gamma_{a_{j+1}})$. In case the minimum did not jump during the first loop (or, equivalently, if ω_j has not been determined during the first loop), then the algorithm executes another loop in which, again, the window is moved rightward; the difference with respect to the first loop is that now the local minimum of $f(x) + g(x)$ at $\gamma_{a_{j+1}}$ is within the window.

In the "Section C – Second loop" of the algorithm, as before, the while loop allows moving, segment-by-segment, the window $[\tau, \theta]$ leftward and, at each step of the while loop, the length ψ of the next rightward movement is determined (row 80) and the new difference δ is computed (rows 82÷92). If $\delta \leq 0$, then a nonfinite or a finite value of ω_j is determined (respectively at rows 106 or 112); otherwise all values and indexes are updated (rows 115÷117) and another step of the loop is executed. In this second loop, the window is moved until its left bound reaches abscissa γ_{b_j}, but the algorithm certainly exits before then.

It is important to observe that when it results $\delta \leq 0$, it is necessary to analyze the shape of $f(x) + g(x)$ in the last part of the window, that is, from $\gamma_{a_{j+1}}$ to θ; as a matter of fact, it is possible that, when the value of $f(x)+g(x)$ at the abscissa $\gamma_{a_{j+1}}$ becomes lower than or equal to all the values in $[\tau, \gamma_{a_{j+1}})$, it is not the global minimum in $[\tau, \theta]$ because a lower value is obtained in $(\gamma_{a_{j+1}}, \theta]$ (such a lower value exists when ϕ, determined at rows 98÷104, is negative); this is the case in which the presence of a local maximum at γ_{b_j} do not produce a jump discontinuity in $x^\circ(t)$, as discussed in the third part of the proof. In this case, ω_j is conventionally set to $+\infty$.

This concludes the proof. □

Lemma 1 is still valid when $f(x) = c \neq 0$ for any $x \leq \gamma_1$. Moreover, Lemma 1 can be easily extended to consider the more general case in which the slope of function $f(x)$ is not null at the beginning, that is, $\mu_0 \neq 0$.

Lemma 2. With reference to the functions $f(x + t)$ and $g(x)$, as considered in Lemma 1, and to the function $x^\circ(t) = \arg\min_x\{f(x + t) + g(x)\}$, $x_1 \leq x \leq x_2$, provided by Lemma 1 itself, the function

$$h(t) = f\big(x^\circ(t) + t\big) + g\big(x^\circ(t)\big) \tag{8}$$

is a continuous, nondecreasing, and piece-wise linear function of the independent variable t, that can be obtained by $f(x + t)$ and $x^\circ(t)$ as follows:

- $h(t)$ is equal to $f(x_2 + t)$ for all t in which $x^\circ(t) = x_2$;
- $h(t)$ is a linear segment with slope ν for all t in which $x^\circ(t)$ decreases with slope -1; the vertical alignments of such segments are such that $h(t)$ is a continuous function;
- $h(t)$ is equal to $f(x_1 + t) + \nu(x_2 - x_1)$ for all t in which $x^\circ(t) = x_1$.

Then:

$$\text{if } Q = 0 \,: \quad h(t) = \begin{cases} f(x_2 + t) & \forall\, t \,:\, x^\circ(t) = x_2 \\ \nu t + \big[f(\gamma_{a_1}) - \nu(\gamma_{a_1} - x_2)\big] & \forall\, t \,:\, x^\circ(t) \neq \{x_1, x_2\} \\ f(x_1 + t) + \nu(x_2 - x_1) & \forall\, t \,:\, x^\circ(t) = x_1 \end{cases} \tag{9a}$$

$$\text{if } Q = 1 : \quad h(t) = \begin{cases} f(x_2 + t) & \forall\, t \,:\, x^\circ(t) = x_2 \\ \nu t + \left[f(\gamma_{a_1}) - \nu(\gamma_{a_1} - x_2) \right] & \forall\, t < t_1^\star : x^\circ(t) \neq \{x_1, x_2\} \\ \nu t + \left[f(\gamma_{a_{l(Q)+1}}) - \nu(\gamma_{a_{l(Q)+1}} - x_2) \right] & \forall\, t \geq t_1^\star : x^\circ(t) \neq \{x_1, x_2\} \\ f(x_1 + t) + \nu(x_2 - x_1) & \forall\, t \,:\, x^\circ(t) = x_1 \end{cases} \tag{9b}$$

$$\text{if } Q > 1 : \quad h(t) = \begin{cases} f(x_2 + t) & \forall\, t \,:\, x^\circ(t) = x_2 \\ \nu t + \left[f(\gamma_{a_1}) - \nu(\gamma_{a_1} - x_2) \right] & \forall\, t < t_1^\star : x^\circ(t) \neq \{x_1, x_2\} \\ \nu t + \left[f(\gamma_{a_{l(q)+1}}) - \nu(\gamma_{a_{l(q)+1}} - x_2) \right] & \forall\, t \in [t_q^\star, t_{q+1}^\star) : x^\circ(t) \neq \{x_1, x_2\} \\ & \qquad q = 1, \ldots, Q-1 \\ \nu t + \left[f(\gamma_{a_{l(Q)+1}}) - \nu(\gamma_{a_{l(Q)+1}} - x_2) \right] & \forall\, t \geq t_Q^\star : x^\circ(t) \neq \{x_1, x_2\} \\ f(x_1 + t) + \nu(x_2 - x_1) & \forall\, t \,:\, x^\circ(t) = x_1 \end{cases} \tag{9c}$$

Proof. When $x = x_2$, it is $g(x) = 0$ (see figure 3); then, when $x^\circ(t) = x_2$, function $h(t) = f(x_2 + t)$. Instead, when $x = x_1$, it is $g(x) = \nu(x_2 - x_1)$ (see again figure 3); then, when $x^\circ(t) = x_1$, function $h(t) = f(x_1 + t) + \nu(x_2 - x_1)$.

When $\gamma_{a_1} - x_2 < t < \gamma_{a_1} - x_1$, with regards to (5a), $x^\circ(t)$ passes linearly (with unitary speed) from the value x_2 at $t = \gamma_{a_1} - x_2$ to the value x_1 at $t = \gamma_{a_1} - x_1$; then, in the same interval, function $h(t)$ passes, with the same dynamics (that is, linearly), from the value $f(x_2 + \gamma_{a_1} - x_2) = f(\gamma_{a_1})$ (at $t = \gamma_{a_1} - x_2$) to the value $f(x_1+\gamma_{a_1}-x_1)+\nu(x_2-x_1) = f(\gamma_{a_1})+\nu(x_2-x_1)$ (at $t = \gamma_{a_1}-x_1$); the segment which joins such values belongs to the line $\nu t+[f(\gamma_{a_1})-\nu(\gamma_{a_1}-x_2)]$. In the same way, when $\gamma_{a_1}-x_2 < t < t_1^\star$, with regards to (5b), $x^\circ(t)$ passes linearly (with unitary speed) from the value x_2 at $t = \gamma_{a_1} - x_2$ to the value $-t_1^\star + \gamma_{a_1}$ at $t = t_1^\star$; then, in the same interval, function $h(t)$ passes linearly from the value $f(\gamma_{a_1})$ (at $t = \gamma_{a_1}-x_2$) to the value $f(\gamma_{a_1})+\nu(x_2+t_1^\star-\gamma_{a_1})$ (at $t = t_1^\star$); the segment which joins such values belongs again to the line $\nu t + [f(\gamma_{a_1}) - \nu(\gamma_{a_1} - x_2)]$. This proves that, when $t : x^\circ(t) \neq \{x_1, x_2\}$ in (9a), and when $t < t_1^\star : x^\circ(t) \neq \{x_1, x_2\}$ in (9b) and (9c), function $h(t) = \nu t + [f(\gamma_{a_1}) - \nu(\gamma_{a_1} - x_2)]$.

In analogous way, when $\gamma_{a_{l(q)+1}} - x_2 < t < \gamma_{a_{l(q)+1}} - x_1$, $q = 1, \ldots, Q$, with regards to (6a) or (7a), $x^\circ(t)$ passes linearly (with unitary speed) from the value x_2 at $t = \gamma_{a_{l(q)+1}} - x_2$ to the value x_1 at $t = \gamma_{a_{l(q)+1}} - x_1$, function $h(t)$ passes linearly from the value $f(\gamma_{a_{l(q)+1}})$ (at $t = \gamma_{a_{l(q)+1}} - x_2$) to the value $f(\gamma_{a_{l(q)+1}}) + \nu(x_2 - x_1)$ (at $t = \gamma_{a_{l(q)+1}}-x_1$), and the segment which joins such values belongs to the line $\nu t+[f(\gamma_{a_{l(q)+1}})-\nu(\gamma_{a_{l(q)+1}}-x_2)]$. In the same way, when $t_q^\star < t < \gamma_{a_{l(q)+1}}-x_1$, $q = 1, \ldots, Q$, with regards to (6b) or (7b), when $\gamma_{a_{l(q)+1}}-x_2 < t < t_{q+1}^\star$, $q = 1, \ldots, Q-1$, with regards to (6c), and when $t_q^\star < t < t_{q+1}^\star$, $q = 1, \ldots, Q-1$, with regards to (6d), function $h(t)$ passes linearly from two values which are connected through the line $\nu t+[f(\gamma_{a_{l(q)+1}})-\nu(\gamma_{a_{l(q)+1}}-x_2)]$. This proves that, when $t \geq t_1^\star : x^\circ(t) \neq \{x_1, x_2\}$ in (9b), when $t \in [t_q^\star, t_{q+1}^\star) : x^\circ(t) \neq \{x_1, x_2\}$, $q = 1, \ldots, Q-1$, in (9c), and when $t \geq t_Q^\star : x^\circ(t) \neq \{x_1, x_2\}$ in (9c), function $h(t) = \nu t + [f(\gamma_{a_{l(q)+1}}) - \nu(\gamma_{a_{l(q)+1}} - x_2)]$, $q = 1, \ldots, Q$. ☐

Note that, when $l(Q) = |A|$ there isn't any $t \geq t_Q^\star$ such that $x^\circ(t) \neq \{x_1, x_2\}$ (since $x_{\mathrm{e}}(t) = x_2$ for any $t \geq t_Q^\star$, in accordance with (7c)). Then, in this case, the term $\nu t + [f(\gamma_{a_1}) - \nu(\gamma_{a_1} - x_2)]$ in (9a) and the term $\nu t+[f(\gamma_{a_{l(Q)+1}})-\nu(\gamma_{a_{l(Q)+1}}-x_2)]$ in (9b) and (9c) have not to be considered as a part of the function $h(t)$.

Lemma 3. Let $f_1(x+t)$ and $f_2(x+t)$ be two continuous nondecreasing piece-wise linear functions of x, parameterized by t, as defined by definition 2. The sum function

$$s(x+t) = f_1(x+t) + f_2(x+t) \tag{10}$$

is still a continuous nondecreasing piece-wise linear functions of x, parameterized by t, which is in accordance with definition 2.

Proof. It is evident that the sum of two continuous piece-wise linear functions of the same argument is a continuous piece-wise linear function of that argument as well; moreover, since all slopes in $f_1(x + t)$ and $f_2(x + t)$ are nonnegative, the slope in a generic segment of $s(x+t)$ is nonnegative as well, because it is the sum of two specific (nonnegative) slopes of $f_1(x + t)$ and $f_2(x + t)$; finally, since the initial slope of both $f_1(x + t)$ and $f_2(x + t)$ is

null, also $s(x+t)$ has initial slope null. Then, $s(x+t)$ is a continuous nondecreasing piece-wise linear functions of x, parameterized by t (which is in accordance with definition 2). This concludes the proof. ☐

Lemma 4. Let $f_1(x)$ and $f_2(x)$ be two continuous nondecreasing piece-wise linear functions of x, as defined by definition 1. The min function

$$m(x) = \min\left\{f_1(x), f_2(x)\right\} \tag{11}$$

is still a continuous nondecreasing piece-wise linear functions of x, which is in accordance with definition 1.

Proof. It is evident that the minimum of two continuous piece-wise linear functions of the same argument is a continuous piece-wise linear function of that argument as well; moreover, since all slopes in $f_1(x)$ and $f_2(x)$ are nonnegative, the slope in a generic segment of $m(x)$ is nonnegative as well, because it corresponds to the slope of one segment of $f_1(x)$ or one segment of $f_2(x)$; finally, since the initial slope of both $f_1(x)$ and $f_2(x)$ is null, also $m(x)$ has initial slope null. Then, $m(x)$ is a continuous nondecreasing piece-wise linear functions of x (which is in accordance with definition 1). This concludes the proof. ☐

4 Examples

4.1 Example 1

Consider the following functions $f(x+t)$ and $g(x)$ (depicted in the same graphic).

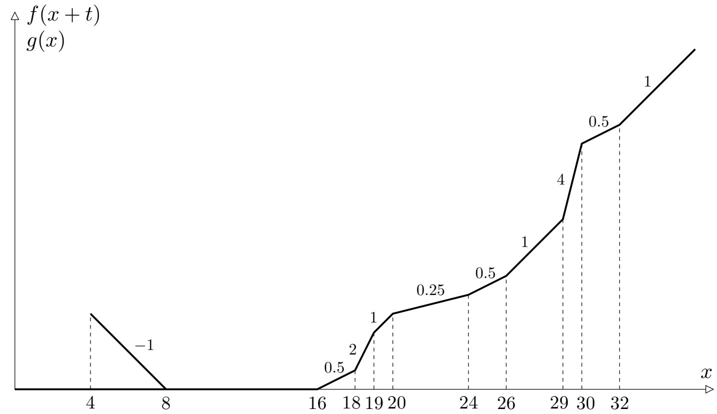


Figure 12: Example 1 – Functions $f(x+t)$ and $g(x)$.

Algorithm 1 provides $\omega_1 = 13.\overline{3}$ and $\omega_2 = 25.\overline{6}$. Then, by applying lemma 1 (taking into account $f(x+t)$, instead of $f(x)$, and $g(x)$) the following function $x^\circ(t)$ is obtained.

$$x^\circ(t) = \begin{cases} x_{\mathrm{s}}(t) & t < 13.\overline{3} \\ x_1(t) & 13.\overline{3} \le t < 25.\overline{6} \\ x_{\mathrm{e}}(t) & t \ge 25.\overline{6} \end{cases}$$

with

$$x_{\mathrm{s}}(t) = \begin{cases} 8 & t < 10 \\ -t+18 & 10 \le t < 13.\overline{3} \end{cases} \qquad x_1(t) = \begin{cases} 8 & 13.\overline{3} \le t < 18 \\ -t+26 & 18 \le t < 22 \\ 4 & 22 \le t < 25.\overline{6} \end{cases}$$

$$x_{\mathrm{e}}(t) = \begin{cases} -t+32 & 25.\overline{6} \le t < 28 \\ 4 & t \ge 28 \end{cases}$$

Note that, $T = \{13.\overline{3}, 25.\overline{6}\}$, that is, $t_1^\star = 13.\overline{3}$ and $t_2^\star = 25.\overline{6}$, and $Q = 2$. Since $T = \Omega$, the mapping function is basically $l(1) = 1$ and $l(2) = 2$. Moreover, $t_1^\star \le \gamma_{a_1} - x_1 = 14$ (then, $x_{\mathrm{s}}(t)$ has the structure of (5b)), $t_1^\star < \gamma_{a_2} - x_2 = 18$ and $t_2^\star > \gamma_{a_2} - x_1 = 22$ (then, $x_1(t)$ has the structure of (6a)), and $t_2^\star \ge \gamma_{a_3} - x_2 = 24$ (then, $x_{\mathrm{e}}(t)$ has the structure of (7b)). The graphical representation of $x^\circ(t)$ is the following.

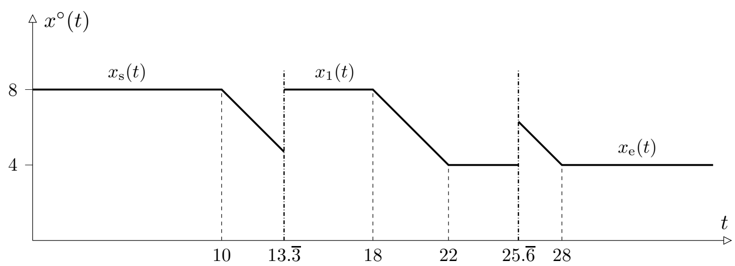


Figure 13: Example 1 – Functions $x^\circ(t)$.

By applying lemma 2 the following function $h(t) = f\big(x^\circ(t)+t\big) + g\big(x^\circ(t)\big)$ is obtained.

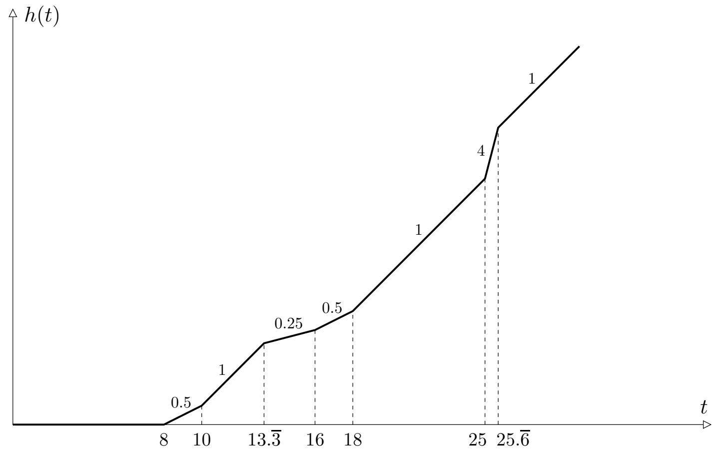


Figure 14: Example 1 – Function $h(t) = f\big(x^\circ(t) + t\big) + g\big(x^\circ(t)\big)$.

In accordance with lemma 2 and, in particular, with (9c), function $h(t)$ is

$$h(t) = \begin{cases} f(8+t) & \forall\, t \,:\, x^\circ(t) = 8 \\ t - 9 & \forall\, t < 13.\overline{3} : x^\circ(t) \neq \{4,8\} \quad (\Rightarrow 10 < t < 13.\overline{3}) \\ t - 12 & \forall\, t \in [13.\overline{3}, 25.\overline{6}) : x^\circ(t) \neq \{4,8\} \quad (\Rightarrow 18 < t < 22) \\ t - 10 & \forall\, t \geq 25.\overline{6} : x^\circ(t) \neq \{4,8\} \quad (\Rightarrow 25.\overline{6} < t < 28) \\ f(4+t) + 4 & \forall\, t \,:\, x^\circ(t) = 4 \end{cases}$$

4.2 Example 2

Consider the following functions $f(x+t)$ and $g(x)$ (depicted in the same graphic).

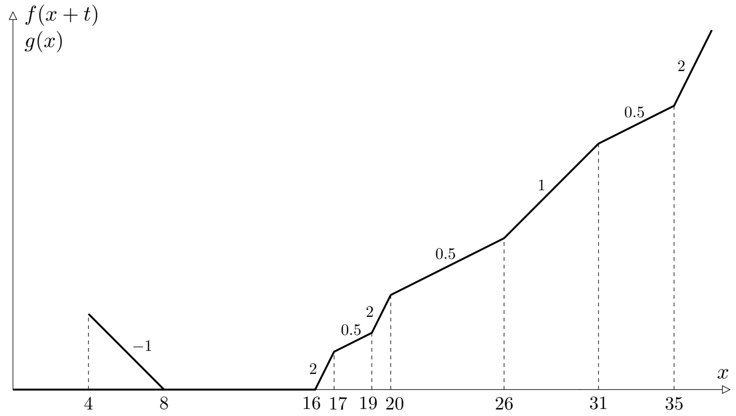


Figure 15: Example 2 – Functions $f(x+t)$ and $g(x)$.

Algorithm 1 provides $\omega_1 = 11$, $\omega_2 = 14$, and $\omega_3 = 23$. Then, by applying lemma 1 (taking into account $f(x+t)$, instead of $f(x)$, and $g(x)$) the following function $x^\circ(t)$ is obtained.

$$x^\circ(t) = \begin{cases} x_\mathrm{s}(t) & t < 11 \\ x_1(t) & 11 \le t < 14 \\ x_2(t) & 14 \le t < 23 \\ x_\mathrm{e}(t) & t \ge 23 \end{cases}$$

with

$$x_\mathrm{s}(t) = \begin{cases} 8 & t < 8 \\ -t+16 & 8 \le t < 11 \end{cases} \qquad x_1(t) = -t + 19$$

$$x_2(t) = \begin{cases} 8 & 14 \le t < 18 \\ -t+26 & 18 \le t < 22 \\ 4 & 22 \le t < 23 \end{cases} \qquad x_\mathrm{e}(t) = \begin{cases} 8 & 23 \le t < 27 \\ -t+35 & 27 \le t < 31 \\ 4 & t \ge 31 \end{cases}$$

Note that, $T = \{11, 14, 23\}$, that is, $t_1^\star = 11$, $t_2^\star = 14$, and $t_3^\star = 23$, and $Q = 3$. Since $T = \Omega$, the mapping function is basically $l(1) = 1$, $l(2) = 2$, and $l(3) = 3$. Moreover, $t_1^\star \le \gamma_{a_1} - x_1 = 12$ (then, $x_\mathrm{s}(t)$ has the structure of (5b)), $t_1^\star \ge \gamma_{a_2} - x_2 = 11$ and $t_2^\star \le \gamma_{a_2} - x_1 = 15$ (then, $x_1(t)$ has the structure of (6d)), $t_2^\star < \gamma_{a_3} - x_2 = 18$ and $t_3^\star > \gamma_{a_3} - x_1 = 22$ (then, $x_2(t)$ has the structure of (6a)), and $t_3^\star < \gamma_{a_4} - x_2 = 27$ (then, $x_\mathrm{e}(t)$ has the structure of (7a)). The graphical representation of $x^\circ(t)$ is the following.

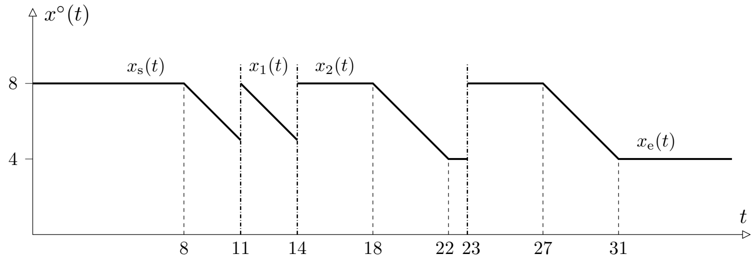


Figure 16: Example 2 – Functions $x^\circ(t)$.

By applying lemma 2 the following function $h(t) = f\big(x^\circ(t) + t\big) + g\big(x^\circ(t)\big)$ is obtained.

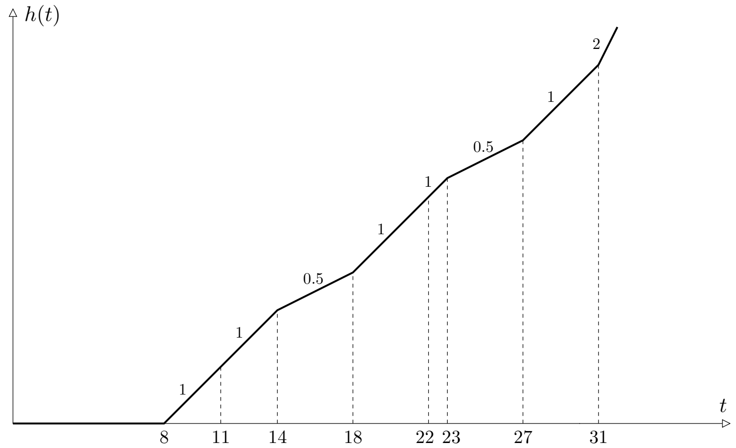


Figure 17: Example 2 – Function $h(t) = f\big(x^{\circ}(t) + t\big) + g\big(x^{\circ}(t)\big)$.

In accordance with lemma 2 and, in particular, with (9c), function $h(t)$ is

$$h(t) = \begin{cases} f(8+t) & \forall\, t\,:\, x^{\circ}(t) = 8 \\ t-8 & \forall\, t < 11 : x^{\circ}(t) \neq \{4,8\} \quad (\Rightarrow 8 < t < 11) \\ t-8 & \forall\, t \in [11,14) : x^{\circ}(t) \neq \{4,8\} \quad (\Rightarrow 11 < t < 14) \\ t-10 & \forall\, t \in [14,23) : x^{\circ}(t) \neq \{4,8\} \quad (\Rightarrow 18 < t < 22) \\ t-12 & \forall\, t \geq 23 : x^{\circ}(t) \neq \{4,8\} \quad (\Rightarrow 27 < t < 31) \\ f(4+t)+4 & \forall\, t\,:\, x^{\circ}(t) = 4 \end{cases}$$

4.3 Example 3

Consider the following functions $f(x+t)$ and $g(x)$ (depicted in the same graphic).

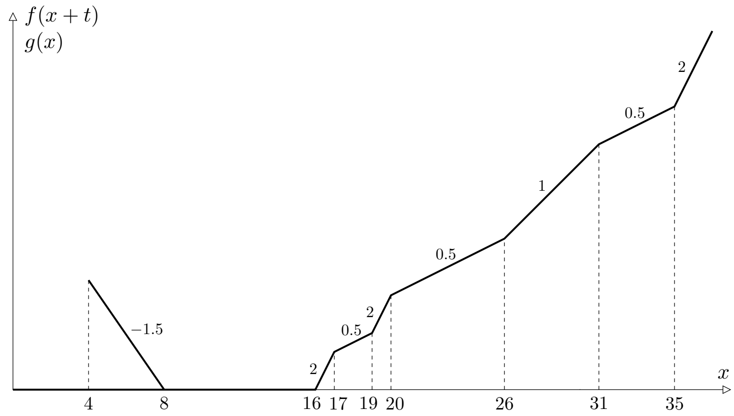


Figure 18: Example 3 – Functions $f(x+t)$ and $g(x)$.

Algorithm 1 provides $\omega_1 = 9.5$ and $\omega_2 = 12.5$. Then, by applying lemma 1 (taking into account $f(x+t)$, instead of $f(x)$, and $g(x)$) the following function $x^\circ(t)$ is obtained.

$$x^\circ(t) = \begin{cases} x_\mathrm{s}(t) & t < 9.5 \\ x_1(t) & 9.5 \le t < 12.5 \\ x_\mathrm{e}(t) & t \ge 12.5 \end{cases}$$

with

$$x_\mathrm{s}(t) = \begin{cases} 8 & t < 8 \\ -t+16 & 8 \le t < 9.5 \end{cases} \qquad x_1(t) = \begin{cases} 8 & 9.5 \le t < 11 \\ -t+19 & 11 \le t < 12.5 \end{cases}$$

$$x_\mathrm{e}(t) = \begin{cases} 8 & 12.5 \le t < 27 \\ -t+35 & 27 \le t < 31 \\ 4 & t \ge 31 \end{cases}$$

Note that, $T = \{9.5, 12.5\}$, that is, $t_1^\star = 9.5$ and $t_2^\star = 12.5$, and $Q = 2$. Since $T = \Omega$, the mapping function is basically $l(1) = 1$ and $l(2) = 2$. Moreover, $t_1^\star \le \gamma_{a_1} - x_1 = 12$ (then, $x_\mathrm{s}(t)$ has the structure of (5b)), $t_1^\star < \gamma_{a_2} - x_2 = 11$ and $t_2^\star \le \gamma_{a_2} - x_1 = 15$ (then, $x_1(t)$ has the structure of (6c)), and $t_2^\star < \gamma_{a_3} - x_2 = 27$ (then, $x_\mathrm{e}(t)$ has the structure of (7a)). The graphical representation of $x^\circ(t)$ is the following.

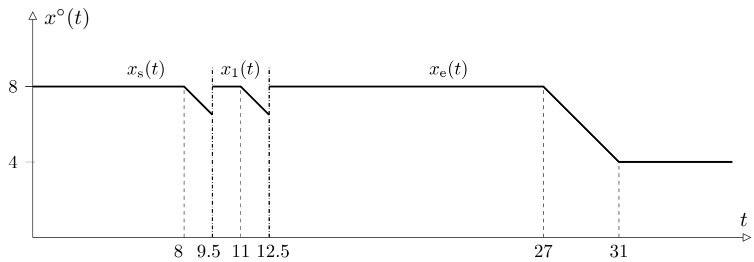


Figure 19: Example 3 – Functions $x^\circ(t)$.

By applying lemma 2 the following function $h(t) = f\big(x^\circ(t)+t\big) + g\big(x^\circ(t)\big)$ is obtained.

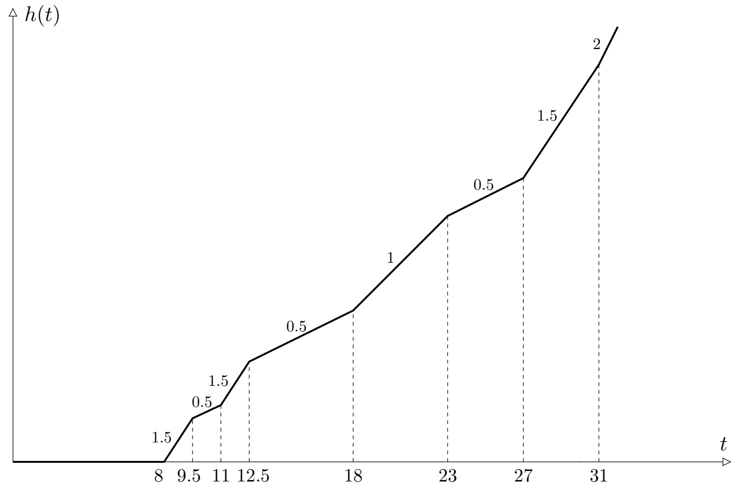


Figure 20: Example 3 – Function $h(t) = f\big(x^{\circ}(t) + t\big) + g\big(x^{\circ}(t)\big)$.

In accordance with lemma 2 and, in particular, with (9c), function $h(t)$ is

$$h(t) = \begin{cases} f(8+t) & \forall\, t \,:\, x^{\circ}(t) = 8 \\ 1.5\,t - 12 & \forall\, t < 9.5 : x^{\circ}(t) \neq \{4, 8\} \quad (\Rightarrow 8 < t < 9.5) \\ 1.5\,t - 13.5 & \forall\, t \in [9.5, 12.5) : x^{\circ}(t) \neq \{4, 8\} \quad (\Rightarrow 11 < t < 12.5) \\ 1.5\,t - 25.5 & \forall\, t \geq 12.5 : x^{\circ}(t) \neq \{4, 8\} \quad (\Rightarrow 27 < t < 31) \\ f(4+t) + 6 & \forall\, t \,:\, x^{\circ}(t) = 4 \end{cases}$$

4.4 Example 4

Consider the following functions $f(x+t)$ and $g(x)$ (depicted in the same graphic).

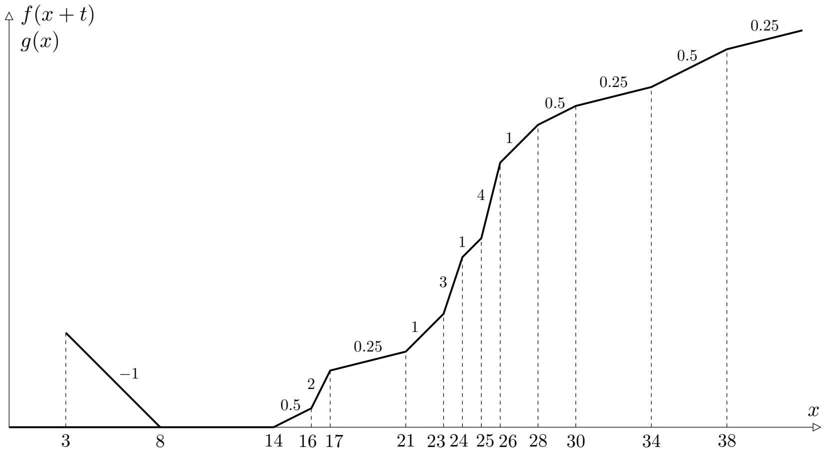


Figure 21: Example 4 – Functions $f(x+t)$ and $g(x)$.

Algorithm 1 provides $\omega_1 = 10.\overline{3}$ and $\omega_2 = 22.5\overline{3}$. Then, by applying lemma 1 (taking into account $f(x+t)$, instead of $f(x)$, and $g(x)$) the following function $x^\circ(t)$ is obtained.

$$x^\circ(t) = \begin{cases} x_{\mathrm{s}}(t) & t < 10.\overline{3} \\ x_1(t) & 10.\overline{3} \le t < 22.5\overline{3} \\ x_{\mathrm{e}}(t) & t \ge 22.5\overline{3} \end{cases}$$

with

$$x_{\mathrm{s}}(t) = \begin{cases} 8 & t < 8 \\ -t+16 & 8 \le t < 10.\overline{3} \end{cases} \qquad x_1(t) = \begin{cases} 8 & 10.\overline{3} \le t < 13 \\ -t+21 & 13 \le t < 18 \\ 3 & 18 \le t < 22.5\overline{3} \end{cases} \qquad x_{\mathrm{e}}(t) = 8$$

Note that, $T = \{10.\overline{3}, 22.5\overline{3}\}$, that is, $t_1^\star = 10.\overline{3}$ and $t_2^\star = 22.5\overline{3}$, and $Q = 2$. Since $T = \Omega$, the mapping function is basically $l(1) = 1$ and $l(2) = 2$. Moreover, $t_1^\star \le \gamma_{a_1} - x_1 = 13$ (then, $x_{\mathrm{s}}(t)$ has the structure of (5b)), $t_1^\star < \gamma_{a_2} - x_2 = 13$ and $t_2^\star > \gamma_{a_2} - x_1 = 18$ (then, $x_1(t)$ has the structure of (6a)); since $l(Q) = |A| = 2$, the function $x_{\mathrm{e}}(t)$ has the structure of (7c). The graphical representation of $x^\circ(t)$ is the following.

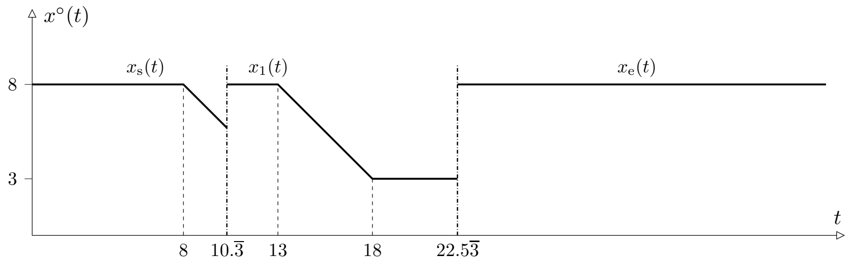


Figure 22: Example 4 – Functions $x^\circ(t)$.

By applying lemma 2 the following function $h(t) = f\big(x^\circ(t) + t\big) + g\big(x^\circ(t)\big)$ is obtained.

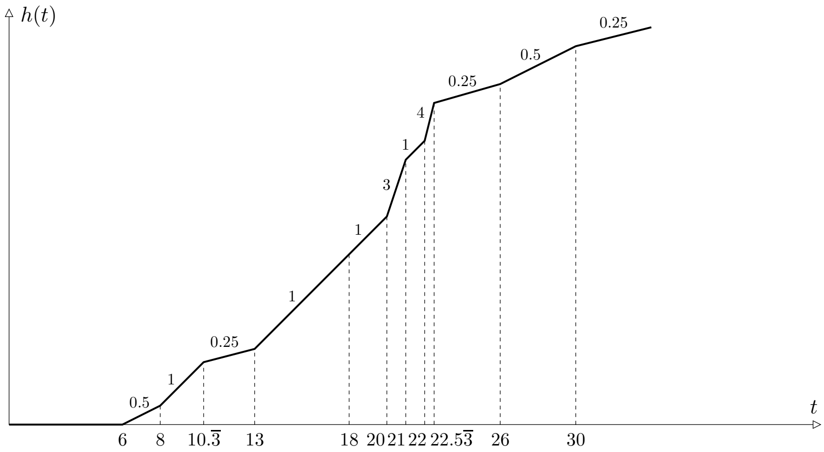


Figure 23: Example 4 – Function $h(t) = f\big(x^{\circ}(t) + t\big) + g\big(x^{\circ}(t)\big)$.

In accordance with lemma 2 and, in particular, with (9c), function $h(t)$ is

$$h(t) = \begin{cases} f(8+t) & \forall\, t \,:\, x^{\circ}(t) = 8 \\ t - 7 & \forall\, t < 10.\overline{3} : x^{\circ}(t) \neq \{3, 8\} \quad (\Rightarrow 8 < t < 10.\overline{3}) \\ t - 9 & \forall\, t \in [10.\overline{3}, 22.5\overline{3}) : x^{\circ}(t) \neq \{3, 8\} \quad (\Rightarrow 13 < t < 18) \\ f(3+t) + 5 & \forall\, t \,:\, x^{\circ}(t) = 3 \end{cases}$$

4.5 Example 5

Consider the following functions $f(x+t)$ and $g(x)$ (depicted in the same graphic).

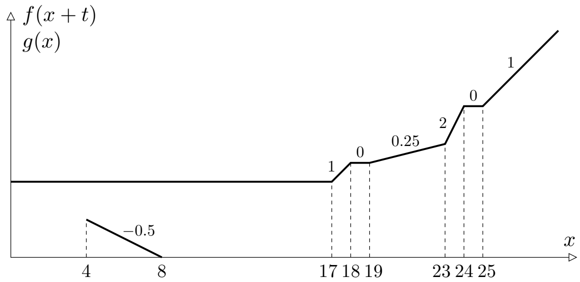


Figure 24: Example 5 – Functions $f(x+t)$ and $g(x)$.

Algorithm 1 provides $\omega_1 = 11$ and $\omega_2 = 19.\overline{6}$. Then, by applying lemma 1 (taking into account $f(x+t)$, instead of $f(x)$, and $g(x)$) the following function $x^\circ(t)$ is obtained.

$$x^\circ(t) = \begin{cases} x_{\mathrm{s}}(t) & t < 11 \\ x_1(t) & 11 \leq t < 19.\overline{6} \\ x_{\mathrm{e}}(t) & t \geq 19.\overline{6} \end{cases}$$

with

$$x_{\mathrm{s}}(t) = \begin{cases} 8 & t < 9 \\ -t + 17 & 9 \leq t < 11 \end{cases} \qquad x_1(t) = \begin{cases} 8 & 11 \leq t < 15 \\ -t + 23 & 15 \leq t < 19 \\ 4 & 19 \leq t < 19.\overline{6} \end{cases}$$

$$x_{\mathrm{e}}(t) = \begin{cases} -t + 25 & 19.\overline{6} \leq t < 21 \\ 4 & t \geq 21 \end{cases}$$

Note that, $T = \{11, 19.\overline{6}\}$, that is, $t_1^\star = 11$ and $t_2^\star = 19.\overline{6}$, and $Q = 2$. Since $T = \Omega$, the mapping function is basically $l(1) = 1$ and $l(2) = 2$. Moreover, $t_1^\star \leq \gamma_{a_1} - x_1 = 13$ (then, $x_{\mathrm{s}}(t)$ has the structure of (5b)), $t_1^\star < \gamma_{a_2} - x_2 = 15$ and $t_2^\star > \gamma_{a_2} - x_1 = 19$ (then, $x_1(t)$ has the structure of (6a)), and $t_2^\star \geq \gamma_{a_3} - x_2 = 17$ (then, $x_{\mathrm{e}}(t)$ has the structure of (7b)). The graphical representation of $x^\circ(t)$ is the following.

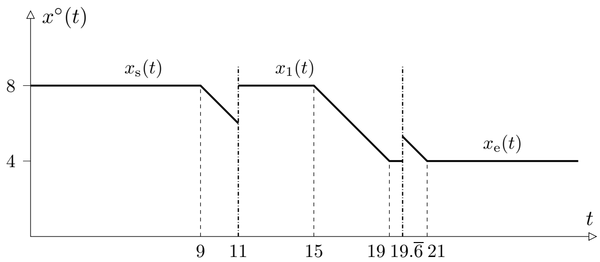


Figure 25: Example 5 – Functions $x^\circ(t)$.

By applying lemma 2 the following function $h(t) = f\big(x^\circ(t) + t\big) + g\big(x^\circ(t)\big)$ is obtained.

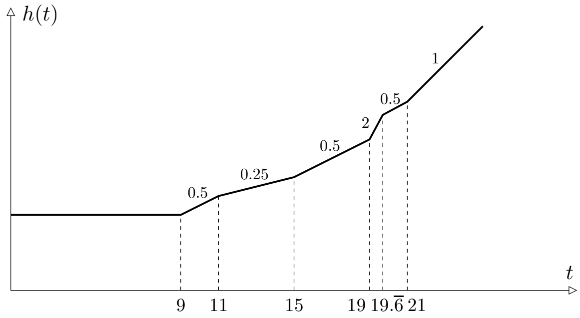


Figure 26: Example 5 – Function $h(t) = f\big(x^\circ(t) + t\big) + g\big(x^\circ(t)\big)$.

In accordance with lemma 2 and, in particular, with (9c), function $h(t)$ is

$$h(t) = \begin{cases} f(8+t) & \forall\, t \,:\, x^\circ(t) = 8 \\ 0.5\,t - 0.5 & \forall\, t < 11 : x^\circ(t) \neq \{4, 8\} \quad (\Rightarrow 9 < t < 11) \\ 0.5\,t - 1.5 & \forall\, t \in [11, 19.\overline{6}) : x^\circ(t) \neq \{4, 8\} \quad (\Rightarrow 15 < t < 19) \\ 0.5\,t - 0.5 & \forall\, t \geq 19.\overline{6} : x^\circ(t) \neq \{4, 8\} \quad (\Rightarrow 19.\overline{6} < t < 21) \\ f(4+t) + 2 & \forall\, t \,:\, x^\circ(t) = 4 \end{cases}$$

4.6 Example 6

Consider the following functions $f(x)$ and $g(x)$ (depicted in the same graphic).

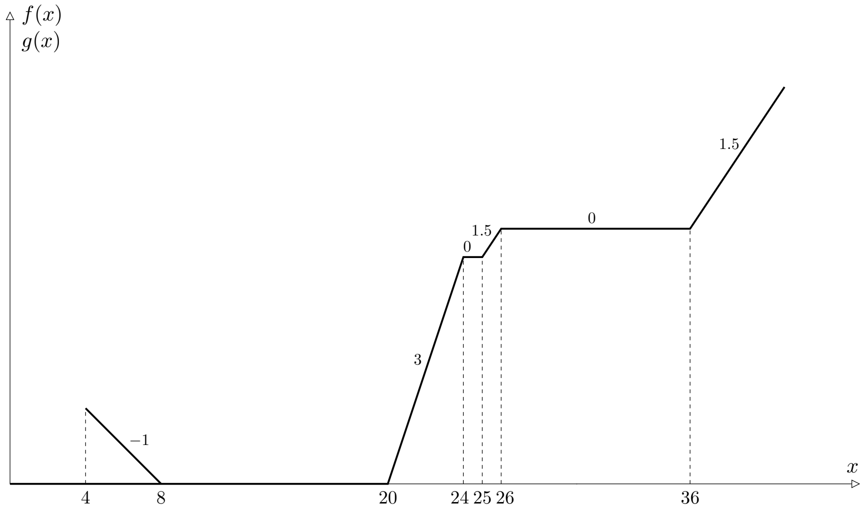


Figure 27: Example 6 – Functions $f(x)$ and $g(x)$.

Algorithm 1 provides $\omega_1 = +\infty$ and $\omega_2 = 19.1\overline{6}$. Then, by applying lemma 1 (taking into account $f(x+t)$, instead of $f(x)$, and $g(x)$) the following function $x^\circ(t)$ is obtained.

$$x^\circ(t) = \begin{cases} x_{\mathrm{s}}(t) & t < 19.1\overline{6} \\ x_{\mathrm{e}}(t) & t \geq 19.1\overline{6} \end{cases}$$

with

$$x_{\mathrm{s}}(t) = \begin{cases} 8 & t < 12 \\ -t+20 & 12 \leq t < 16 \\ 4 & 16 \leq t < 19.1\overline{6} \end{cases} \qquad x_{\mathrm{e}}(t) = \begin{cases} 8 & 19.1\overline{6} \leq t < 28 \\ -t+36 & 28 \leq t < 32 \\ 4 & t \geq 32 \end{cases}$$

Note that, $T = \{19.1\overline{6}\}$, that is, $t_1^\star = 19.1\overline{6}$, and $Q = 1$. The mapping function provides $l(1) = 2$. Moreover, $t_1^\star > \gamma_{a_1} - x_1 = 16$ (then, $x_{\mathrm{s}}(t)$ has the structure of (5a)) and $t_1^\star < \gamma_{a_3} - x_2 = 28$ (then, $x_{\mathrm{e}}(t)$ has the structure of (7a)). The graphical representation of $x^\circ(t)$ is the following.

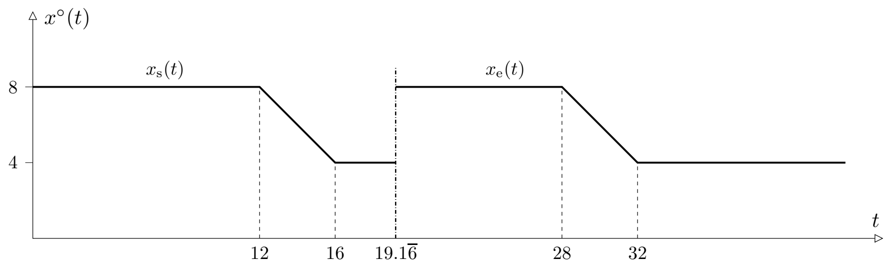


Figure 28: Example 6 – Functions $x^\circ(t)$.

By applying lemma 2 the following function $h(t) = f\big(x^\circ(t)+t\big) + g\big(x^\circ(t)\big)$ is obtained.

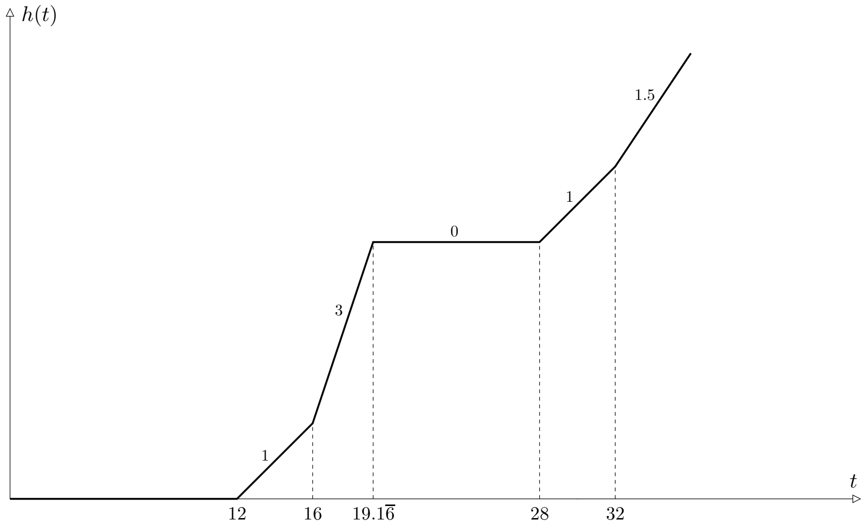


Figure 29: Example 6 – Function $h(t) = f\big(x^{\circ}(t) + t\big) + g\big(x^{\circ}(t)\big)$.

In accordance with lemma 2 and, in particular, with (9b), function $h(t)$ is

$$h(t) = \begin{cases} f(8+t) & \forall\, t \,:\, x^{\circ}(t) = 8 \\ t - 12 & \forall\, t < 19.1\overline{6} : x^{\circ}(t) \neq \{4, 8\} \quad (\Rightarrow 12 < t < 16) \\ t - 14.5 & \forall\, t \geq 19.1\overline{6} : x^{\circ}(t) \neq \{4, 8\} \quad (\Rightarrow 28 < t < 32) \\ f(4+t) + 4 & \forall\, t \,:\, x^{\circ}(t) = 4 \end{cases}$$

4.7 Example 7

Consider the following functions $f(x)$ and $g(x)$ (depicted in the same graphic).

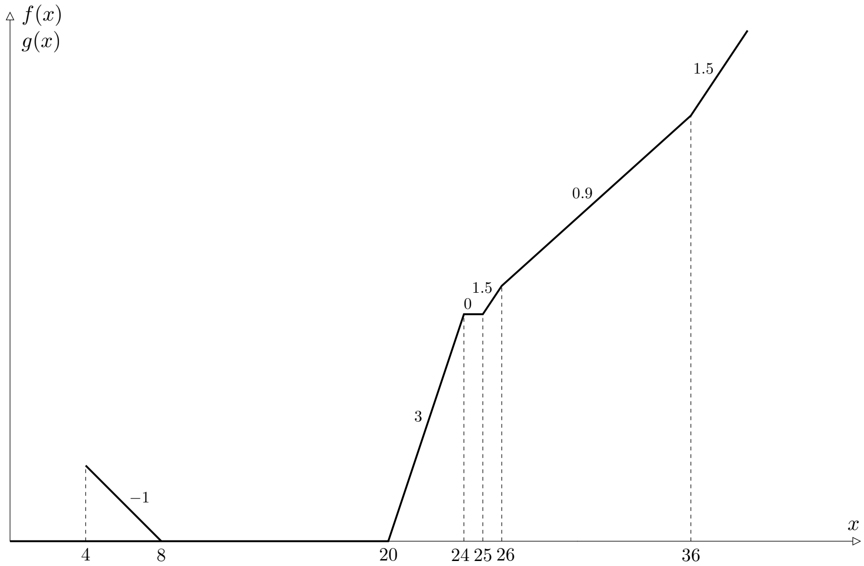


Figure 30: Example 7 – Functions $f(x)$ and $g(x)$.

Algorithm 1 provides $\omega_1 = 19.5$ and $\omega_2 = 21.\overline{3}$. Then, by applying lemma 1 (taking into account $f(x+t)$, instead of $f(x)$, and $g(x)$) the following function $x^{\circ}(t)$ is obtained.

$$x^{\circ}(t) = \begin{cases} x_{\mathrm{s}}(t) & t < 19.5 \\ x_1(t) & 19.5 \leq t < 21.\overline{3} \\ x_{\mathrm{e}}(t) & t \geq 21.\overline{3} \end{cases}$$

with

$$x_{\mathrm{s}}(t) = \begin{cases} 8 & t < 12 \\ -t+20 & 12 \leq t < 16 \\ 4 & 16 \leq t < 19.5 \end{cases} \qquad x_1(t) = \begin{cases} -t+25 & 19.5 \leq t < 21 \\ 4 & 21 \leq t < 21.\overline{3} \end{cases}$$

$$x_{\mathrm{e}}(t) = \begin{cases} 8 & 21.\overline{3} \leq t < 28 \\ -t+36 & 28 \leq t < 32 \\ 4 & t \geq 32 \end{cases}$$

Note that, $T = \{19.5, 21.\overline{3}\}$, that is, $t_1^\star = 19.5$ and $t_2^\star = 21.\overline{3}$, and $Q = 2$. Since $T = \Omega$, the mapping function is basically $l(1) = 1$ and $l(2) = 2$. Moreover, $t_1^\star > \gamma_{a_1} - x_1 = 16$ (then, $x_{\mathrm{s}}(t)$ has the structure of (5a)), $t_1^\star \geq \gamma_{a_2} - x_2 = 17$ and $t_2^\star > \gamma_{a_2} - x_1 = 21$ (then, $x_1(t)$ has the structure of (6b)), and $t_2^\star < \gamma_{a_4} - x_2 = 28$ (then, $x_{\mathrm{e}}(t)$ has the structure of (7a)). The graphical representation of $x^{\circ}(t)$ is the following.

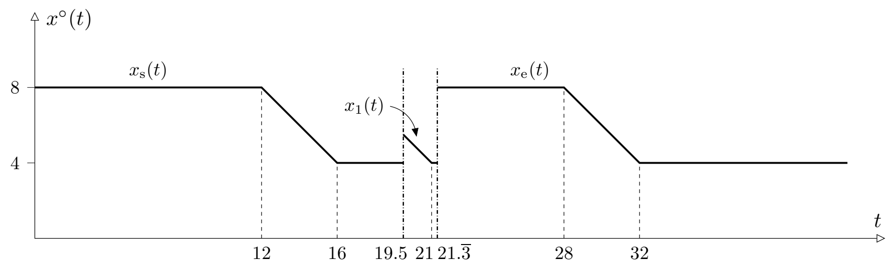


Figure 31: Example 7 – Functions $x^{\circ}(t)$.

By applying lemma 2 the following function $h(t) = f\big(x^{\circ}(t) + t\big) + g\big(x^{\circ}(t)\big)$ is obtained.

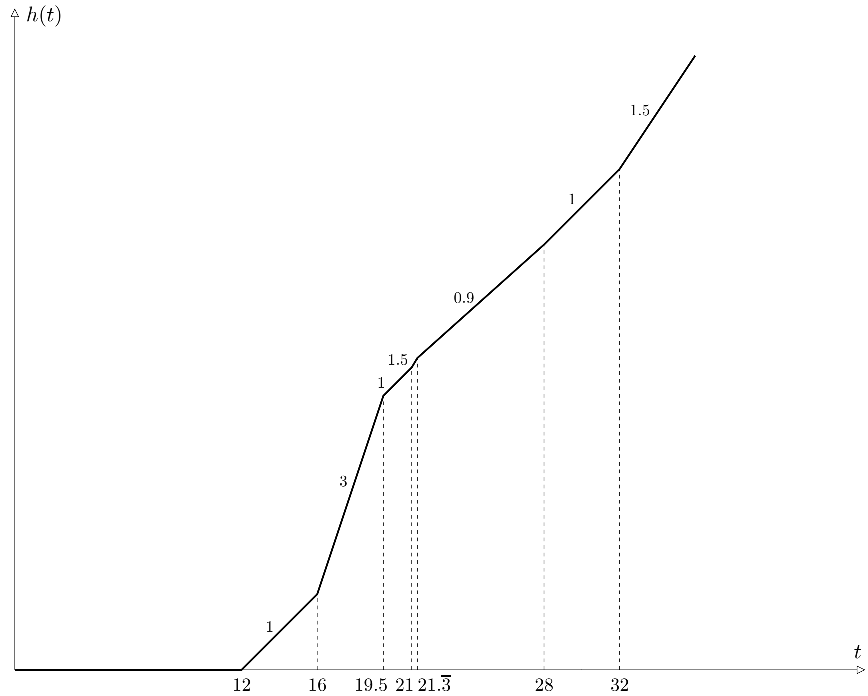


Figure 32: Example 7 – Function $h(t) = f\big(x^\circ(t) + t\big) + g\big(x^\circ(t)\big)$.

In accordance with lemma 2 and, in particular, with (9c), function $h(t)$ is

$$h(t) = \begin{cases} f(8+t) & \forall\, t\,:\, x^\circ(t) = 8 \\ t - 12 & \forall\, t < 19.5 : x^\circ(t) \neq \{4, 8\} \quad (\Rightarrow 12 < t < 16) \\ t - 5 & \forall\, t \in [19.5, 21.\overline{3}) : x^\circ(t) \neq \{4, 8\} \quad (\Rightarrow 19.5 < t < 21) \\ t - 5.5 & \forall\, t \geq 21.\overline{3} : x^\circ(t) \neq \{4, 8\} \quad (\Rightarrow 28 < t < 32) \\ f(4+t) + 4 & \forall\, t\,:\, x^\circ(t) = 4 \end{cases}$$

4.8 Example 8

Consider the following functions $f(x)$ and $g(x)$ (depicted in the same graphic).

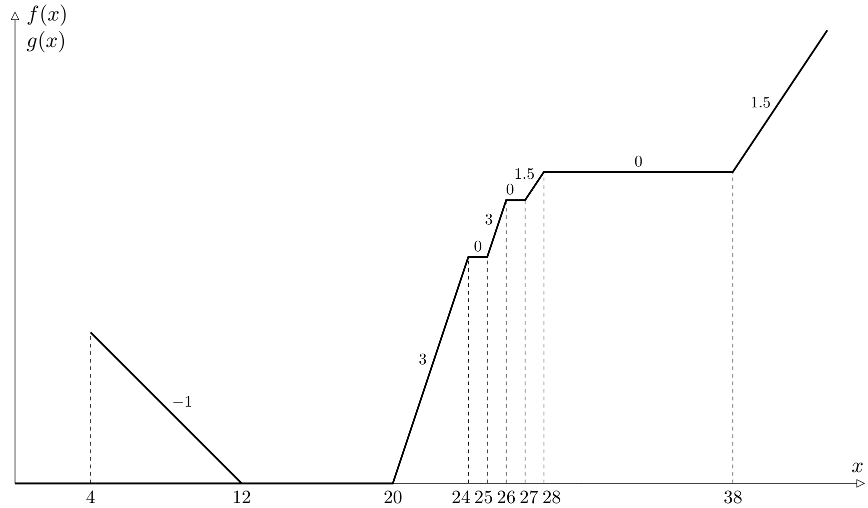


Figure 33: Example 8 – Functions $f(x)$ and $g(x)$.

Algorithm 1 provides $\omega_1 = +\infty$, $\omega_2 = +\infty$, and $\omega_3 = 18.8\overline{3}$. Then, by applying lemma 1 (taking into account $f(x+t)$, instead of $f(x)$, and $g(x)$) the following function $x^\circ(t)$ is obtained.

$$x^\circ(t) = \begin{cases} x_\mathrm{s}(t) & t < 18.8\overline{3} \\ x_\mathrm{e}(t) & t \geq 18.8\overline{3} \end{cases}$$

with

$$x_\mathrm{s}(t) = \begin{cases} 12 & t < 8 \\ -t+20 & 8 \leq t < 16 \\ 4 & 16 \leq t < 18.8\overline{3} \end{cases} \qquad x_\mathrm{e}(t) = \begin{cases} 12 & 18.8\overline{3} \leq t < 26 \\ -t+38 & 26 \leq t < 34 \\ 4 & t \geq 34 \end{cases}$$

Note that, $T = \{18.8\overline{3}\}$, that is, $t_1^\star = 18.8\overline{3}$, and $Q = 1$. The mapping function provides $l(1) = 3$. Moreover, $t_1^\star > \gamma_{a_1} - x_1 = 16$ (then, $x_\mathrm{s}(t)$ has the structure of (5a)) and $t_1^\star < \gamma_{a_4} - x_2 = 26$ (then, $x_\mathrm{e}(t)$ has the structure of (7a)). The graphical representation of $x^\circ(t)$ is the following.

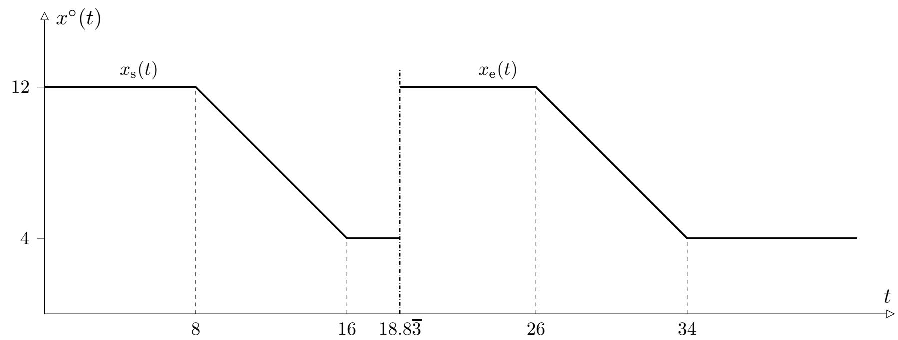


Figure 34: Example 8 – Functions $x^\circ(t)$.

By applying lemma 2 the following function $h(t) = f\big(x^\circ(t)+t\big) + g\big(x^\circ(t)\big)$ is obtained.

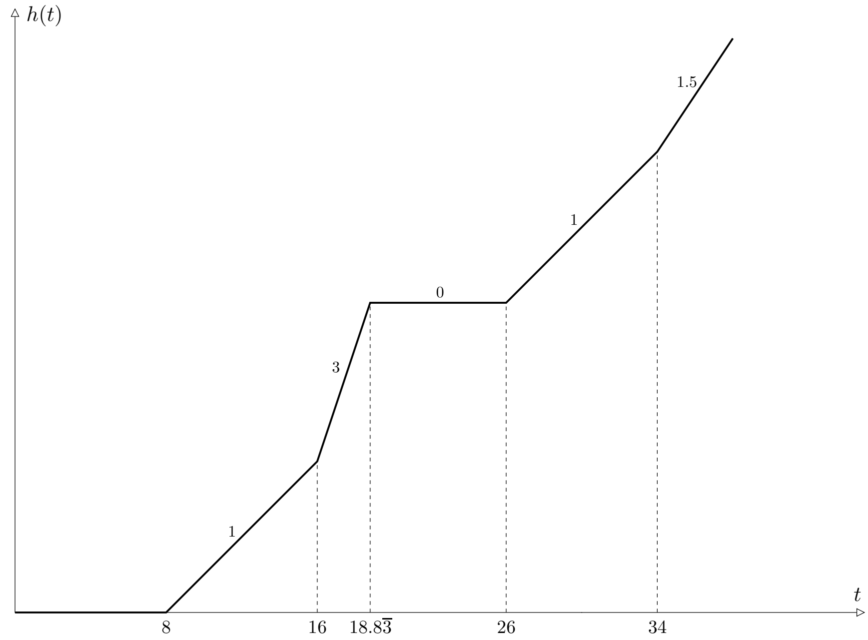


Figure 35: Example 8 – Function $h(t) = f\big(x^\circ(t) + t\big) + g\big(x^\circ(t)\big)$.

In accordance with lemma 2 and, in particular, with (9b), function $h(t)$ is

$$h(t) = \begin{cases} f(12+t) & \forall\, t \,:\, x^\circ(t) = 12 \\ t - 8 & \forall\, t < 18.8\overline{3} : x^\circ(t) \neq \{4, 12\} \quad (\Rightarrow 8 < t < 16) \\ t - 9.5 & \forall\, t \geq 18.8\overline{3} : x^\circ(t) \neq \{4, 12\} \quad (\Rightarrow 26 < t < 34) \\ f(4+t) + 8 & \forall\, t \,:\, x^\circ(t) = 4 \end{cases}$$

4.9 Example 9

Consider the following functions $f(x)$ and $g(x)$ (depicted in the same graphic).

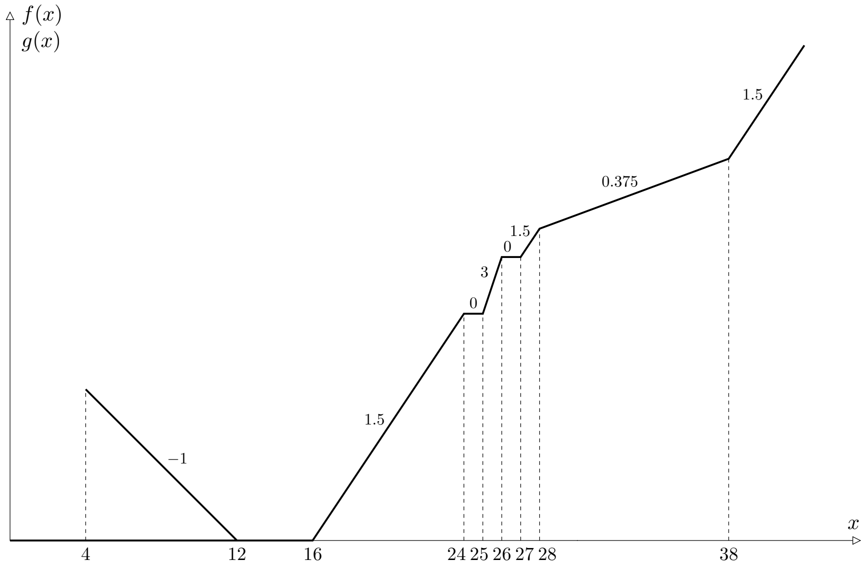


Figure 36: Example 9 – Functions $f(x)$ and $g(x)$.

Algorithm 1 provides $\omega_1 = 18$, $\omega_2 = +\infty$, and $\omega_3 = 18.4$ (the application of algorithm 1 is reported in the following, for each value of j). Then, by applying lemma 1 (taking into account $f(x+t)$, instead of $f(x)$, and $g(x)$) the following function $x^\circ(t)$ is obtained.

$$x^\circ(t) = \begin{cases} x_\mathrm{s}(t) & t < 18 \\ x_1(t) & 18 \le t < 18.4 \\ x_\mathrm{e}(t) & t \ge 18.4 \end{cases}$$

with

$$x_\mathrm{s}(t) = \begin{cases} 12 & t < 4 \\ -t+16 & 4 \le t < 12 \\ 4 & 12 \le t < 18 \end{cases} \qquad x_1(t) = -t + 25 \qquad x_\mathrm{e}(t) = \begin{cases} 12 & 18.4 \le t < 26 \\ -t+38 & 26 \le t < 34 \\ 4 & t \ge 34 \end{cases}$$

Note that, $T = \{18, 18.4\}$, that is, $t_1^\star = 18$ and $t_2^\star = 18.4$, and $Q = 2$. The mapping function provides $l(1) = 1$ and $l(2) = 3$. Moreover, $t_1^\star > \gamma_{a_1} - x_1 = 12$ (then, $x_\mathrm{s}(t)$ has the structure of (5a)), $t_1^\star \ge \gamma_{a_2} - x_2 = 13$ and $t_2^\star \le \gamma_{a_2} - x_1 = 21$ (then, $x_1(t)$ has the structure of (6d)), and $t_2^\star < \gamma_{a_4} - x_2 = 26$ (then, $x_\mathrm{e}(t)$ has the structure of (7a)). The graphical representation of $x^\circ(t)$ is the following.

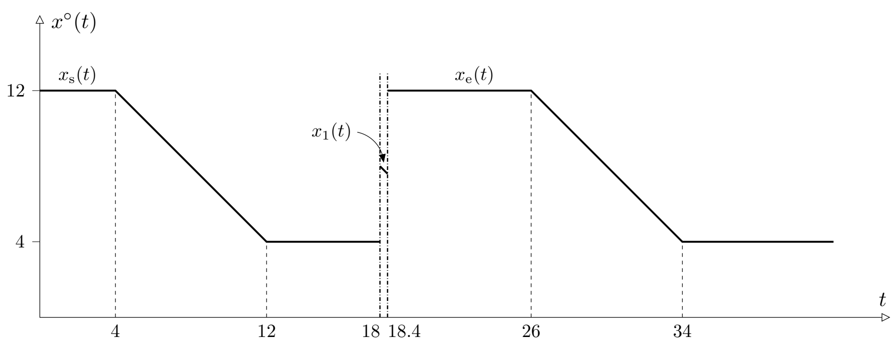


Figure 37: Example 9 – Functions $x^\circ(t)$.

By applying lemma 2 the following function $h(t) = f\big(x^\circ(t) + t\big) + g\big(x^\circ(t)\big)$ is obtained.

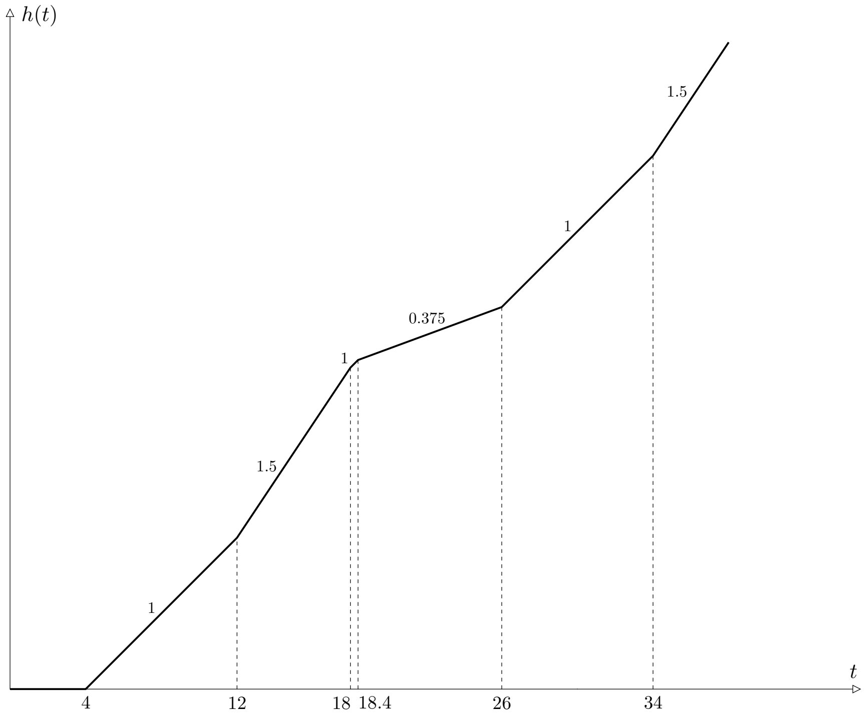


Figure 38: Example 9 – Function $h(t) = f\big(x^{\circ}(t) + t\big) + g\big(x^{\circ}(t)\big)$.

In accordance with lemma 2 and, in particular, with (9c), function $h(t)$ is

$$h(t) = \begin{cases} f(12+t) & \forall\, t\, :\, x^{\circ}(t) = 12 \\ t-4 & \forall\, t < 18 : x^{\circ}(t) \neq \{4, 12\} \quad (\Rightarrow 4 < t < 12) \\ t-1 & \forall\, t \in [18, 18.4) : x^{\circ}(t) \neq \{4, 12\} \quad (\Rightarrow 18 < t < 18.4) \\ t-5.75 & \forall\, t \geq 18.4 : x^{\circ}(t) \neq \{4, 12\} \quad (\Rightarrow 26 < t < 34) \\ f(4+t)+8 & \forall\, t\, :\, x^{\circ}(t) = 4 \end{cases}$$

5 Application to the single machine scheduling

Consider a single machine scheduling problem in which 1 job of class P_1 and 2 jobs of class P_2 must be executed. The due dates, the marginal tardiness costs of jobs, the processing time bounds and the marginal deviation costs of jobs are the:

$\alpha_{1,1} = 0.5$	$dd_{1,1} = 10$
$\beta_1 = 1$	
$pt_1^{\text{low}} = 1$	$pt_1^{\text{nom}} = 4$

$\alpha_{2,1} = 0.25$	$dd_{2,1} = 12$
$\alpha_{2,2} = 0.75$	$dd_{2,2} = 20$
$\beta_2 = 1$	
$pt_2^{\text{low}} = 1$	$pt_2^{\text{nom}} = 2$

No setup is required between the execution of jobs of different classes. The evolution of the system state can be represented by the following diagram.

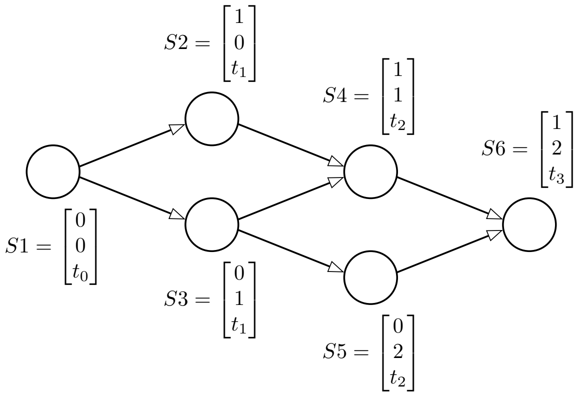
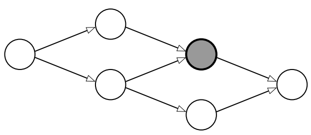


Figure 39: State diagram in the case of two classes of jobs, where $N_1 = 1$ and $N_2 = 2$.

The application of dynamic programming, in conjunction with the new lemmas, provides the following optimal control strategies.

Stage 2 – State $[1\ 1\ t_2]^T$



In state $[1\ 1\ t_2]^T$ the unique job of class P_1 has been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuos) decision variable τ only (which corresponds to the processing time $pt_{2,2}$), is

$$\alpha_{2,2}\ \max\{t_2 + \tau - dd_{2,2}\ ,\ 0\} + \beta_2\,(pt_2^{\text{nom}} - \tau) + J^\circ_{1,2}(t_3)$$

that can be written as $f(pt_{2,2} + t_2) + g(pt_{2,2})$ being

$$f(pt_{2,2} + t_2) = 0.75 \cdot \max\{pt_{2,2} + t_2 - 20\ ,\ 0\}$$

$$g(pt_{2,2}) = \begin{cases} 2 - pt_{2,2} & pt_{2,2} \in [1,2) \\ 0 & pt_{2,2} \notin [1,2) \end{cases}$$

the two functions illustrated in figure 40.

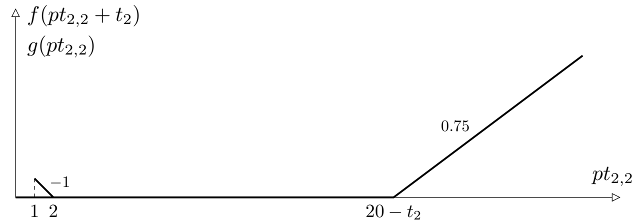


Figure 40: Functions $f(pt_{2,2}+t_2)$ and $g(pt_{2,2})$ in state $[1\ 1\ t_2]^T$.

It is possible to apply lemma 1 (note that $f(pt_{2,2}+t_2)$ follows definition 2 and $g(pt_{2,2})$ follows definition 3), which provides the optimal processing time

$$pt^\circ_{2,2}(t_2) = \arg \min_{\substack{pt_{2,2} \\ 1 \le pt_{2,2} \le 2}} \big\{ f(pt_{2,2}+t_2) + g(pt_{2,2}) \big\} = x_\mathrm{e}(t_2)$$

illustrated in figure 41, being $x_\mathrm{e}(t_2)$ the function

$$x_\mathrm{e}(t_2) = 2$$

$pt^\circ_{2,2}(t_2)$ and $x_\mathrm{e}(t_2)$ are in accordance with (4a) and (7c), respectively. Note that, in this case, $A=\emptyset$, $B=\emptyset$, $|A|=|B|=0$, and then there is no need of executing algorithm 1. Taking into account the mandatory decision about the class of the next job to be executed; the optimal control strategies for this state are

$$\delta^\circ_1(1,1,t_2) = 0 \quad \forall\, t_2 \qquad \delta^\circ_2(1,1,t_2) = 1 \quad \forall\, t_2$$

$$\tau^\circ(1,1,t_2) = pt^\circ_{2,2}(t_2) = 2 \quad \forall\, t_2$$

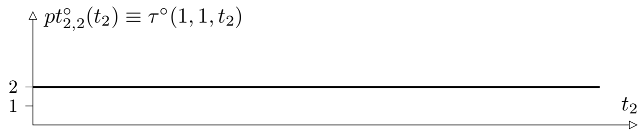


Figure 41: Optimal control strategy $\tau^\circ(1,1,t_2)$ (service time) in state $[1\ 1\ t_2]^T$.

The optimal cost-to-go

$$J^\circ_{1,1}(t_2) = f\big(pt^\circ_{2,2}(t_2)+t_2\big) + g\big(pt^\circ_{2,2}(t_2)\big)$$

illustrated in figure 42, is provided by lemma 2.

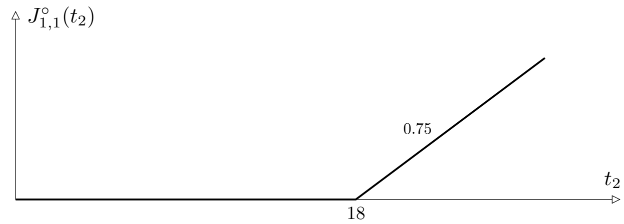
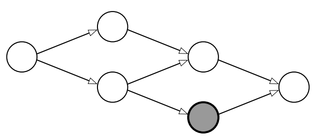


Figure 42: Optimal cost-to-go $J^\circ_{1,1}(t_2)$ in state $[1\ 1\ t_2]^T$.

Stage 2 – State $[0\ 2\ t_2]^T$



In state $[0\ 2\ t_2]^T$ all jobs of class P_2 have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuos) decision variable τ only (which corresponds to the processing time $pt_{1,1}$), is

$$\alpha_{1,1}\ \max\{t_2+\tau-dd_{1,1}\,,\,0\} + \beta_1\,(pt^{\mathrm{nom}}_1-\tau) + J^\circ_{1,2}(t_3)$$

that can be written as $f(pt_{1,1} + t_2) + g(pt_{1,1})$ being

$$f(pt_{1,1} + t_2) = 0.5 \cdot \max\{pt_{1,1} - 10\,,\, 0\}$$

$$g(pt_{1,1}) = \begin{cases} 4 - pt_{1,1} & pt_{1,1} \in [1,4) \\ 0 & pt_{1,1} \notin [1,4) \end{cases}$$

the two functions illustrated in figure 43.

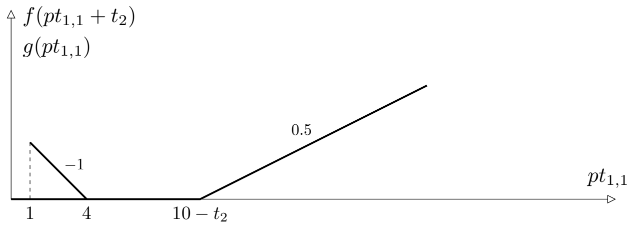


Figure 43: Functions $f(pt_{1,1} + t_2)$ and $g(pt_{1,1})$ in state $[0\ 2\ t_2]^T$.

It is possible to apply lemma 1 (note that $f(pt_{1,1}, t_2)$ follows definition 2 and $g(pt_{1,1})$ follows definition 3), which provides the optimal processing time

$$pt^\circ_{1,1}(t_2) = \arg \min_{\substack{pt_{1,1} \\ 1 \le pt_{1,1} \le 4}} \left\{ f(pt_{1,1} + t_2) + g(pt_{1,1}) \right\} = x_e(t_2)$$

illustrated in figure 44, being $x_e(t_2)$ the function

$$x_e(t_2) = 4$$

$pt^\circ_{1,1}(t_2)$ and $x_e(t_2)$ are in accordance with (4a) and (7c), respectively. Note that, in this case, $A = \emptyset$, $B = \emptyset$, $|A| = |B| = 0$, and then there is no need of executing algorithm 1. Taking into account the mandatory decision about the class of the next job to be executed; the optimal control strategies for this state are

$$\delta^\circ_1(0, 2, t_2) = 1 \quad \forall\, t_2 \qquad \delta^\circ_2(0, 2, t_2) = 0 \quad \forall\, t_2$$

$$\tau^\circ(0, 2, t_2) = pt^\circ_{1,1}(t_2) = 4 \quad \forall\, t_2$$

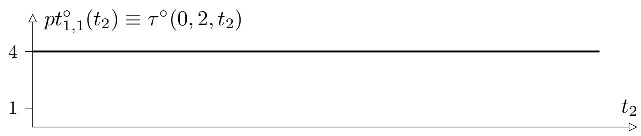


Figure 44: Optimal control strategy $\tau^\circ(0, 2, t_2)$ (service time) in state $[0\ 2\ t_2]^T$.

The optimal cost-to-go

$$J^\circ_{0,2}(t_2) = f\big(pt^\circ_{1,1}(t_2) + t_2\big) + g\big(pt^\circ_{1,1}(t_2)\big)$$

illustrated in figure 45, is provided by lemma 2.

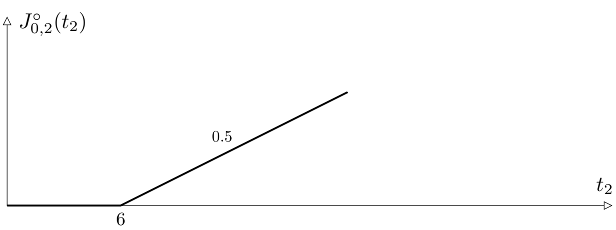
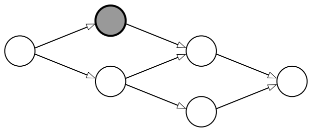


Figure 45: Optimal cost-to-go $J^\circ_{0,2}(t_2)$ in state $[0\ 2\ t_2]^T$.

Stage 1 – State $[1\ 0\ t_1]^T$



In state $[1\ 0\ t_1]^T$ the unique job of class P_1 has been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuos) decision variable τ only (which corresponds to the processing time $pt_{2,1}$), is

$$\alpha_{2,1}\ \max\{t_1 + \tau - dd_{2,1}\ ,\ 0\} + (pt_2^{\mathrm{nom}} - \tau) + J^\circ_{1,1}(t_2)$$

that can be written as $f(pt_{2,1} + t_1) + g(pt_{2,1})$ being

$$f(pt_{2,1} + t_1) = 0.25 \cdot \max\{pt_{2,1} + t_1 - 12\ ,\ 0\} + J^\circ_{1,1}(pt_{2,1} + t_1)$$

$$g(pt_{2,1}) = \begin{cases} 2 - pt_{2,1} & pt_{2,1} \in [1, 2) \\ 0 & pt_{2,1} \notin [1, 2) \end{cases}$$

the two functions illustrated in figure 46.

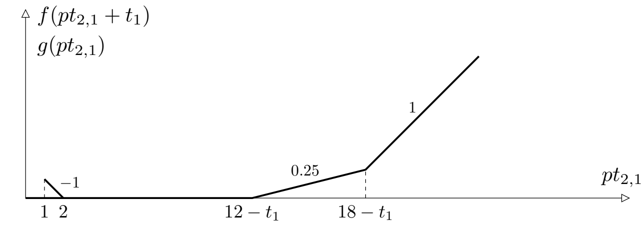


Figure 46: Functions $f(pt_{2,1} + t_1)$ and $g_{2,1}(pt_{2,1})$ in state $[1\ 0\ t_1]^T$.

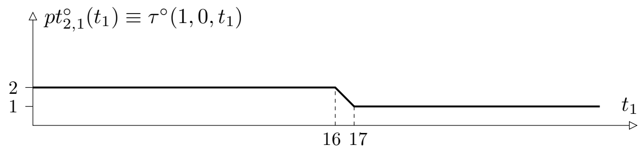


Figure 47: Optimal control strategy $\tau^\circ(1, 0, t_1)$ (service time) in state $[1\ 0\ t_1]^T$.

It is possible to apply lemma 1 (note that $f(pt_{2,1} + t_1)$ follows definition 2 and $g(pt_{2,1})$ follows definition 3), which provides the optimal processing time

$$pt^\circ_{2,1}(t_1) = \arg \min_{\substack{pt_{2,1} \\ 1 \le pt_{2,1} \le 2}} \big\{ f(pt_{2,1} + t_1) + g(pt_{2,1}) \big\} = x_{\mathrm{e}}(t_1)$$

illustrated in figure 47, being $x_{\mathrm{e}}(t_1)$ the function

$$x_{\mathrm{e}}(t_1) = \begin{cases} 2 & t_1 < 16 \\ -t_1 + 18 & 16 \le t_1 < 17 \\ 1 & t_1 \ge 17 \end{cases}$$

$pt^\circ_{2,1}(t_1)$ and $x_{\mathrm{e}}(t_1)$ are in accordance with (4a) and (7a), respectively. Note that, in this case, $A = \{2\}$, $|A| = 1$, $\gamma_{a_1} = 18$; moreover, since $B = \emptyset$ and $|B| = 0$, there is no need of executing algorithm 1. It is worth again remarking that, in the current state, the decision about the class of the next job to be executed is mandatory, since the unique job of class P_1 has been completed. Then,

$$\delta^\circ_1(1, 0, t_1) = 0 \quad \forall\, t_1 \qquad \delta^\circ_2(1, 0, t_1) = 1 \quad \forall\, t_1$$

$$\tau^\circ(1, 0, t_1) = pt^\circ_{2,1}(t_1) = \begin{cases} 2 & t_1 < 16 \\ -t_1 + 18 & 16 \le t_1 < 17 \\ 1 & t_1 \ge 17 \end{cases}$$

The optimal cost-to-go

$$J^{\circ}_{1,0}(t_1) = f\big(pt^{\circ}_{2,1}(t_1) + t_1\big) + g\big(pt^{\circ}_{2,1}(t_1)\big)$$

illustrated in figure 48, is provided by lemma 2.

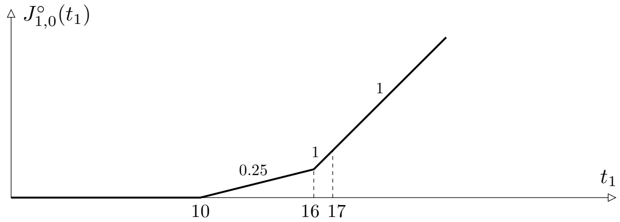
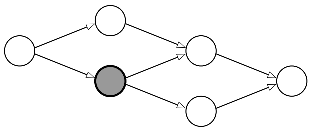


Figure 48: Optimal cost-to-go $J^{\circ}_{1,0}(t_1)$ in state $[1\ 0\ t_1]^T$.

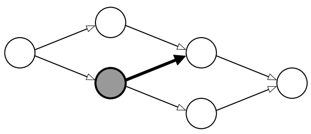
Stage 1 – State $[0\ 1\ t_1]^T$



In state $[0\ 1\ t_1]^T$, the cost function to be minimized, with respect to the (continuos) decision variable τ and to the (binary) decision variables δ_1 and δ_2 is

$$\delta_1\big[\alpha_{1,1}\ \max\{t_1 + \tau - dd_{1,1}\ ,\ 0\} + \beta_1\,(pt_1^{\mathrm{nom}} - \tau) + J^{\circ}_{1,1}(t_2)\big] +$$
$$+\ \delta_2\big[\alpha_{2,2}\ \max\{t_1 + \tau - dd_{2,2}\ ,\ 0\} + \beta_2\,(pt_2^{\mathrm{nom}} - \tau) + J^{\circ}_{0,2}(t_2)\big]$$

Case i) in which it is assumed $\delta_1 = 1$ (and $\delta_2 = 0$).



In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{1,1}$, the following function

$$\alpha_{1,1}\ \max\{t_1 + \tau - dd_{1,1}\ ,\ 0\} + \beta_1\,(pt_1^{\mathrm{nom}} - \tau) + J^{\circ}_{1,1}(t_2)$$

that can be written as $f(pt_{1,1} + t_1) + g(pt_{1,1})$ being

$$f(pt_{1,1} + t_1) = 0.5 \cdot \max\{pt_{1,1} + t_1 - 10\,,\ 0\} + J^{\circ}_{1,1}(pt_{1,1} + t_1)$$

$$g(pt_{1,1}) = \begin{cases} 4 - pt_{1,1} & pt_{1,1} \in [1,4) \\ 0 & pt_{1,1} \notin [1,4) \end{cases}$$

the two functions illustrated in figure 49.

It is possible to apply lemma 1 (note that $f(pt_{1,1} + t_1)$ follows definition 2 and $g(pt_{1,1})$ follows definition 3), which provides the function

$$pt^{\circ}_{1,1}(t_1) = \arg \min_{\substack{pt_{1,1} \\ 1 \le pt_{1,1} \le 4}} \big\{f(pt_{1,1} + t_1) + g(pt_{1,1})\big\} = x_{\mathrm{e}}(t_1)$$

illustrated in figure 50, being $x_{\mathrm{e}}(t_1)$ the function

$$x_{\mathrm{e}}(t_1) = \begin{cases} 4 & t_1 < 14 \\ -t_1 + 18 & 14 \le t_1 < 17 \\ 1 & t_1 \ge 17 \end{cases}$$

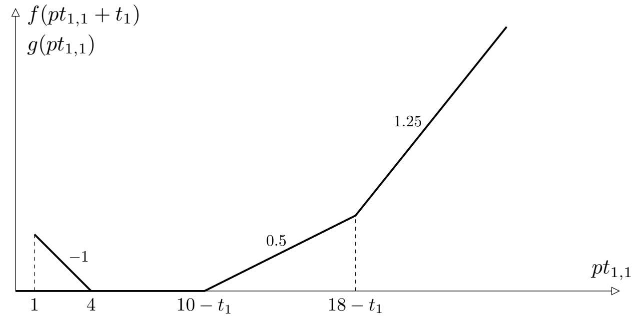


Figure 49: Functions $f(pt_{1,1}+t_1)$ and $g(pt_{1,1})$ in state $[0\ 1\ t_1]^T$.

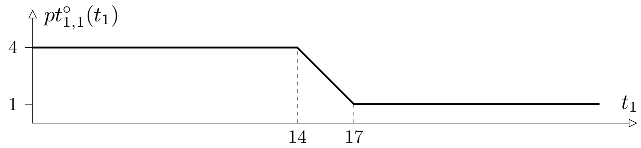


Figure 50: Function $pt^\circ_{1,1}(t_1)$.

$pt^\circ_{1,1}(t_1)$ and $x_{\mathrm{e}}(t_1)$ are in accordance with (4a) and (7a), respectively. Note that, in this case, $A=\{2\}$, $|A|=1$, $\gamma_{a_1}=18$; moreover, since $B=\emptyset$ and $|B|=0$, there is no need of executing algorithm 1.

The conditioned cost-to-go

$$J^\circ_{0,1}(t_1 \mid \delta_1=1) = f\big(pt^\circ_{1,1}(t_1)+t_1\big) + g\big(pt^\circ_{1,1}(t_1)\big)$$

illustrated in figure 51, is provided by lemma 2.

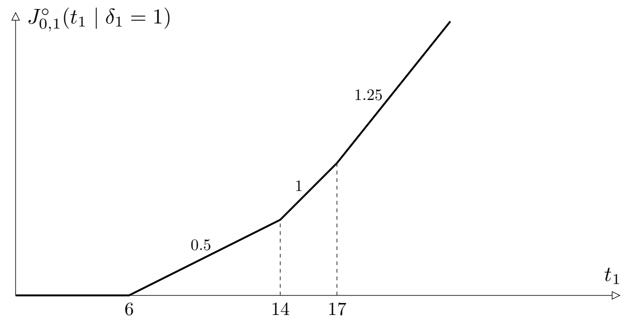
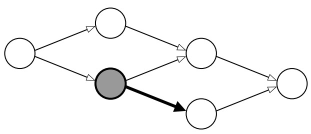


Figure 51: Conditioned cost-to-go $J^\circ_{0,1}(t_1 \mid \delta_1=1)$.

Case ii) in which it is assumed $\delta_2=1$ (and $\delta_1=0$).



In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{2,2}$, the following function

$$\alpha_{2,2}\ \max\{t_1+\tau-dd_{2,2}\,,\,0\} + \beta_2\,(pt_2^{\mathrm{nom}}-\tau) + J^\circ_{0,2}(t_2)$$

that can be written as $f(pt_{2,2}+t_1)+g(pt_{2,2})$ being

$$f(pt_{2,2}+t_1) = 0.75\cdot\max\{pt_{2,2}+t_1-20\,,\,0\} + J^\circ_{0,2}(pt_{2,2}+t_1)$$

$$g(pt_{2,2}) = \begin{cases} 2-pt_{2,2} & pt_{2,2}\in[1,2) \\ 0 & pt_{2,2}\notin[1,2) \end{cases}$$

the two functions illustrated in figure 52.

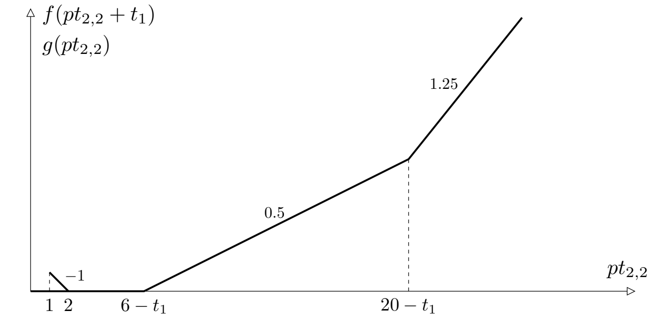


Figure 52: Functions $f(pt_{2,2}+t_1)$ and $g(pt_{2,2})$ in state $[0\ 1\ t_1]^T$.

It is possible to apply lemma 1 (note that $f(pt_{2,2}+t_1)$ follows definition 2 and $g(pt_{2,2})$ follows definition 3), which provides the function

$$pt^\circ_{2,2}(t_1) = \arg \min_{\substack{pt_{2,2} \\ 1 \le pt_{2,2} \le 2}} \left\{ f(pt_{2,2}+t_1) + g(pt_{2,2}) \right\} = x_{\mathrm{e}}(t_1)$$

illustrated in figure 53, being $x_{\mathrm{e}}(t_1)$ the function

$$x_{\mathrm{e}}(t_1) = \begin{cases} 2 & t_1 < 18 \\ -t_1 + 20 & 18 \le t_1 < 19 \\ 1 & t_1 \ge 19 \end{cases}$$

$pt^\circ_{2,2}(t_1)$ and $x_{\mathrm{e}}(t_1)$ are in accordance with (4a) and (7a), respectively. Note that, in this case, $A = \{2\}$, $|A| = 1$, $\gamma_{a_1} = 20$; moreover, since $B = \emptyset$ and $|B| = 0$, there is no need of executing algorithm 1.

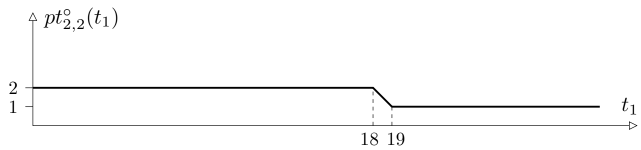


Figure 53: Function $pt^\circ_{2,2}(t_1)$.

The conditioned cost-to-go

$$J^\circ_{0,1}(t_1 \mid \delta_2 = 1) = f\big(pt^\circ_{2,2}(t_1) + t_1\big) + g\big(pt^\circ_{2,2}(t_1)\big)$$

illustrated in figure 54, is provided by lemma 2.

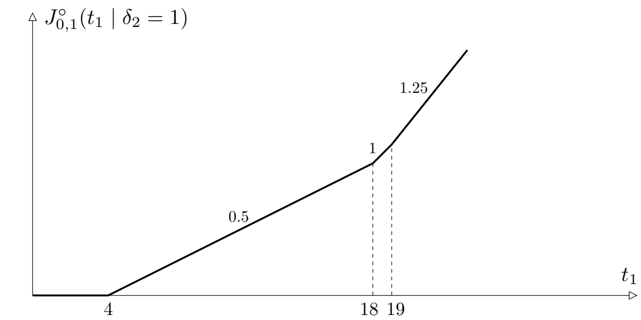


Figure 54: Conditioned cost-to-go $J^\circ_{0,1}(t_1 \mid \delta_2 = 1)$.

In order to find the optimal cost-to-go $J^\circ_{0,1}(t_1)$, it is necessary to carry out the following minimization

$$J^\circ_{0,1}(t_1) = \min \left\{ J^\circ_{0,1}(t_1 \mid \delta_1 = 1)\,,\ J^\circ_{0,1}(t_1 \mid \delta_2 = 1) \right\}$$

which provides, in accordance with lemma 4, the function illustrated in figure 55.

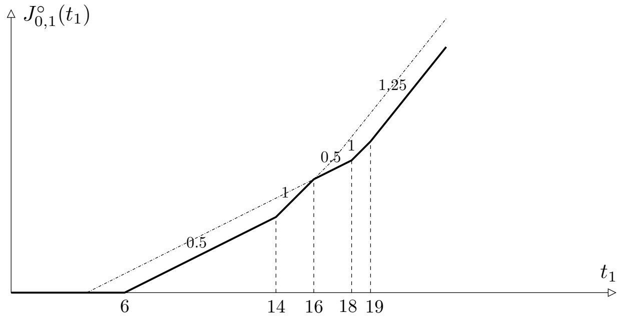


Figure 55: Optimal cost-to-go $J^\circ_{0,1}(t_1)$ in state $[0\ 1\ t_1]^T$.

Since $J^\circ_{0,1}(t_1 \mid \delta_1 = 1)$ is the minimum in $(-\infty, 16)$ and $J^\circ_{0,1}(t_1 \mid \delta_2 = 1)$ is the minimum in $[16, +\infty)$, the optimal control strategies for this state are

$$\delta^\circ_1(0,1,t_1) = \begin{cases} 1 & t_1 < 16 \\ 0 & t_1 \geq 16 \end{cases} \qquad \delta^\circ_2(0,1,t_1) = \begin{cases} 0 & t_1 < 16 \\ 1 & t_1 \geq 16 \end{cases}$$

$$\tau^\circ(0,1,t_1) = \delta^\circ_1(0,1,t_1)\, pt^\circ_{1,1}(t_1) + \delta^\circ_2(0,1,t_1)\, pt^\circ_{2,2}(t_1) = \begin{cases} 4 & t_1 < 14 \\ -t_1 + 18 & 14 \leq t_1 < 16 \\ 2 & 16 \leq t_1 < 18 \\ -t_1 + 20 & 18 \leq t_1 < 19 \\ 1 & t_1 \geq 19 \end{cases}$$

illustrated in figures 56, 57, and 58, respectively.

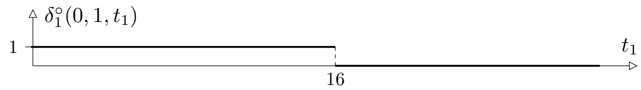


Figure 56: Optimal control strategy $\delta^\circ_1(0,1,t_1)$ in state $[0\ 1\ t_1]^T$.

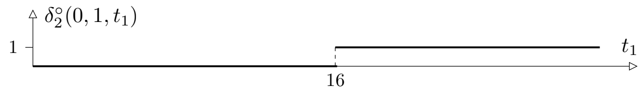


Figure 57: Optimal control strategy $\delta^\circ_2(0,1,t_1)$ in state $[0\ 1\ t_1]^T$.

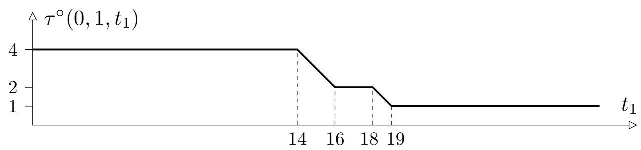
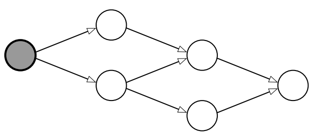


Figure 58: Optimal control strategy $\tau^\circ(0,1,t_1)$ (service time) in state $[0\ 1\ t_1]^T$.

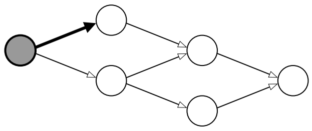
Stage 0 – State $[0\ 0\ t_0]^T$ (initial state)



In state $[0\ 0\ t_0]^T$, the cost function to be minimized, with respect to the (continuos) decision variable τ and to the (binary) decision variables δ_1 and δ_2 is

$$\delta_1\big[\alpha_{1,1}\ \max\{t_0 + \tau - dd_{1,1}\ ,\ 0\} + \beta_1\,(pt_1^{\mathrm{nom}} - \tau) + J^\circ_{1,0}(t_1)\big] +$$
$$+\ \delta_2\big[\alpha_{2,1}\ \max\{t_0 + \tau - dd_{2,1}\ ,\ 0\} + \beta_2\,(pt_2^{\mathrm{nom}} - \tau) + J^\circ_{0,1}(t_1)\big]$$

Case i) in which it is assumed $\delta_1 = 1$ (and $\delta_2 = 0$).



In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{1,1}$, the following function

$$\alpha_{1,1}\ \max\{t_0 + \tau - dd_{1,1}\ ,\ 0\} + \beta_1\,(pt_1^{\text{nom}} - \tau) + J^\circ_{1,0}(t_1)$$

that can be written as $f(pt_{1,1} + t_0) + g(pt_{1,1})$ being

$$f(pt_{1,1} + t_0) = 0.5 \cdot \max\{pt_{1,1} + t_0 - 10\ ,\ 0\} + J^\circ_{1,1}(pt_{1,1} + t_0)$$

$$g(pt_{1,1}) = \begin{cases} 4 - pt_{1,1} & pt_{1,1} \in [1, 4) \\ 0 & pt_{1,1} \notin [1, 4) \end{cases}$$

the two functions illustrated in figure 59.

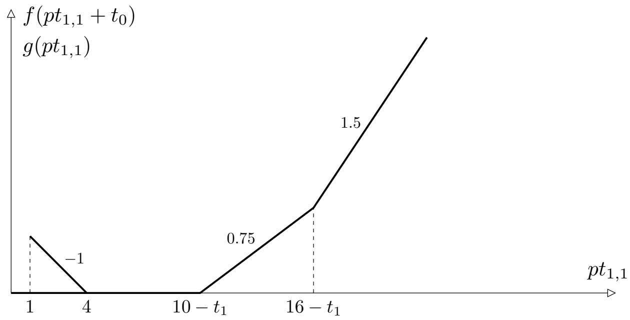


Figure 59: Functions $f(pt_{1,1} + t_0)$ and $g(pt_{1,1})$ in state $[0\ 0\ t_0]^T$.

It is possible to apply lemma 1 (note that $f(pt_{1,1} + t_0)$ follows definition 2 and $g(pt_{1,1})$ follows definition 3), which provides the function

$$pt^\circ_{1,1}(t_0) = \arg \min_{\substack{pt_{1,1} \\ 1 \le pt_{1,1} \le 4}} \left\{ f(pt_{1,1} + t_0) + g(pt_{1,1}) \right\} = x_{\text{e}}(t_0)$$

illustrated in figure 60, being $x_{\text{e}}(t_0)$ the function

$$x_{\text{e}}(t_0) = \begin{cases} 4 & t_1 < 12 \\ -t_1 + 16 & 12 \le t_1 < 15 \\ 1 & t_1 \ge 15 \end{cases}$$

$pt^\circ_{1,1}(t_0)$ and $x_{\text{e}}(t_0)$ are in accordance with (4a) and (7a), respectively. Note that, in this case, $A = \{2\}$, $|A| = 1$, $\gamma_{a_1} = 16$; moreover, since $B = \emptyset$ and $|B| = 0$, there is no need of executing algorithm 1.

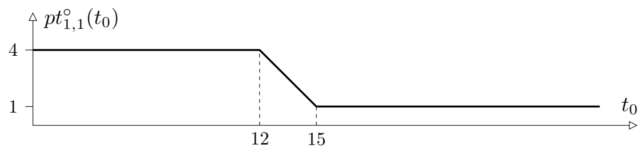


Figure 60: Function $pt^\circ_{1,1}(t_0)$.

The conditioned cost-to-go

$$J^\circ_{0,0}(t_0 \mid \delta_1 = 1) = f\big(pt^\circ_{1,1}(t_0) + t_0\big) + g\big(pt^\circ_{1,1}(t_0)\big)$$

illustrated in figure 61, is provided by lemma 2.

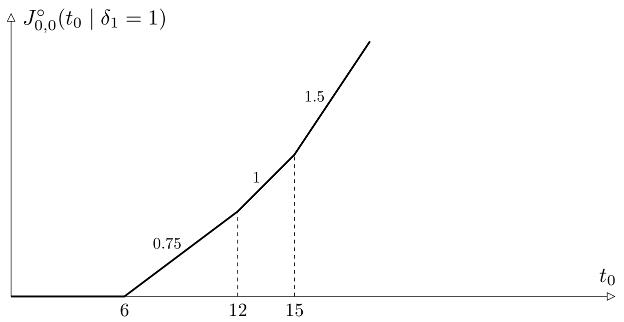
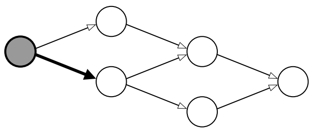


Figure 61: Conditioned cost-to-go $J^{\circ}_{0,0}(t_0 \mid \delta_1 = 1)$.

Case ii) in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).



In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{2,1}$, the following function

$$\alpha_{2,1}\ \max\{t_0 + \tau - dd_{2,1}\,,\, 0\} + \beta_2\,(pt_2^{\mathrm{nom}} - \tau) + J^{\circ}_{0,1}(t_1)$$

that can be written as $f(pt_{2,1} + t_0) + g(pt_{2,1})$ being

$$f(pt_{2,1} + t_0) = 0.25 \cdot \max\{pt_{2,1} + t_0 - 12\,,\, 0\} + J^{\circ}_{0,1}(pt_{2,1} + t_0)$$

$$g(pt_{2,1}) = \begin{cases} 2 - pt_{2,1} & pt_{2,1} \in [1,2) \\ 0 & pt_{2,1} \notin [1,2) \end{cases}$$

the two functions illustrated in figure 62.

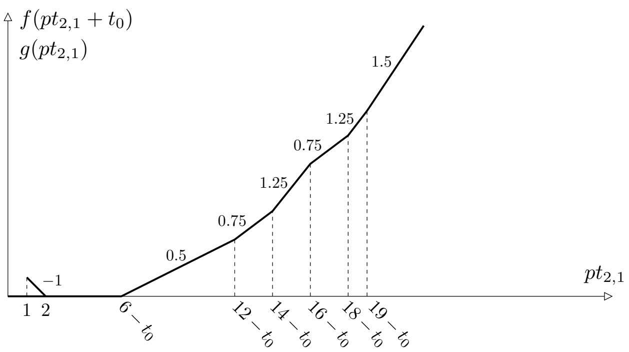


Figure 62: Functions $f(pt_{2,1} + t_0)$ and $g(pt_{2,1})$ in state $[0\ 0\ t_0]^T$.

It is possible to apply lemma 1 (note that $f(pt_{2,1} + t_0)$ follows definition 2 and $g(pt_{2,1})$ follows definition 3), which provides the function

$$pt^{\circ}_{2,1}(t_0) = \arg \min_{\substack{pt_{2,1} \\ 1 \le pt_{2,1} \le 2}} \left\{ f(pt_{2,1} + t_0) + g(pt_{2,1}) \right\} = \begin{cases} x_{\mathrm{s}}(t_0) & t_0 < 14.5 \\ x_{\mathrm{e}}(t_0) & t_0 \ge 14.5 \end{cases}$$

illustrated in figure 63, in which 14.5 is the value ω_1 determined by applying algorithm 1, and being $x_{\mathrm{s}}(t_0)$ and $x_{\mathrm{e}}(t_0)$ the functions

$$x_{\mathrm{s}}(t_0) = \begin{cases} 2 & t_0 < 12 \\ -t_0 + 14 & 12 \le t_0 < 13 \\ 1 & 13 \le t_0 < 14.5 \end{cases}$$

$$x_{\mathrm{e}}(t_0) = \begin{cases} 2 & 14.5 \le t_0 < 16 \\ -t_0 + 18 & 16 \le t_0 < 17 \\ 1 & t_0 \ge 17 \end{cases}$$

$pt^{\circ}_{2,1}(t_0)$, $x_{\mathrm{s}}(t_0)$, and $x_{\mathrm{e}}(t_0)$ are in accordance with (4b), (5a), and (7a), respectively.

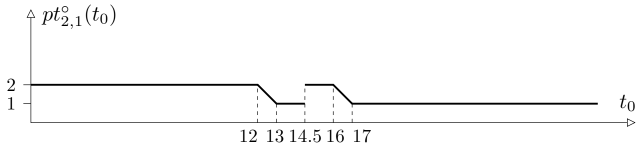


Figure 63: Function $pt^{\circ}_{2,1}(t_0)$.

ω_1 is determined by applying algorithm 1 as follows.

$A = \{3, 5\}$	$\lvert A\rvert = 2$	$\gamma_{a_1} = 14$	$\gamma_{a_2} = 18$	$x_1 = 1$	$x_2 = 2$
$B = \{4\}$	$\lvert B\rvert = 1$	$\gamma_{b_1} = 16$			

With $j = 1$, the “Section A – Initialization” part of the algorithm provides:

[row 1]: $\gamma_0 = -\infty$

[row 2]: $h \ge 0 \ : \ \gamma_h \le 16 - (2 - 1) < \gamma_{h+1} \quad \Rightarrow \quad h = 3$

[row 3]: $i = 4$

[row 4]: $\gamma_7 = +\infty$

[row 5]: $k \le 6 \ : \ \gamma_k < 16 + (2 - 1) \le \gamma_{k+1} \quad \Rightarrow \quad k = 4$

[row 6]: condition: $j = |B|$ and $|A| = |B|$ ($1 = 1$ and $2 = 1$) is false

[row 10]: $\begin{cases} \tilde{\mu}_3 = 1.25 - 1 = 0.25 \\ \tilde{\mu}_4 = 0.75 - 1 = -0.25 \end{cases}$

[row 12]: $\tau = 16 - (2 - 1) = 15$

[row 13]: $\theta = 16$

[row 14]: $d = \max\{0, 0.25 \cdot (16 - 15)\} = 0.25$

[row 15]: condition: $h < b_j - 1$ ($3 < 4 - 1$) is false

[row 20]: $\lambda = 3$

[row 21]: $\xi = 4$

Since condition: $h < b_1$ and $i < a_2$ ($4 < 5$ and $3 < 4$) [row 22] is true, the “Section B – First Loop” part of the algorithm is executed:

[row 23]: $\psi = \min\{16 - 15, 18 - 16\} = 1$

[row 24]: condition: $\gamma_{h+1} - \tau \le \gamma_{i+1} - \theta$ ($16 - 15 \le 18 - 16$) is true

[row 25]: $\lambda = 3 + 1 = 4$

[row 27]: condition: $\gamma_{h+1} - \tau \ge \gamma_{i+1} - \theta$ ($16 - 15 \ge 18 - 16$) is false

[row 30]: $\delta = \max\{0, -0.25 \cdot \big[18 - (15 + 1)\big] = 0$

[row 31]: condition: $\lambda < b_j - 1$ ($3 < 4 - 1$) is false

[row 36]: condition: $\xi = b_j$ ($4 = 4$) is true

[row 37]: $\delta = 0 - 0.25 \cdot \big[(16 + 1) - 16\big] = -0.25$

[row 43]: condition: $\delta \le 0$ ($-0.25 \le 0$) is true

[row 44]: $a_0 = 0$

[row 45]: $r \geq 1 \;:\; a_{r-1} \leq 3 < a_r \quad \Rightarrow \quad r = 2$

[row 46]: condition: $r \leq j$ $(2 \leq 1)$ is false

[row 68]: $\omega_1 = 15 - 1 + \dfrac{0.25}{0.25 + 0.25} = 14.5$

[row 69]: exit algorithm

The conditioned cost-to-go

$$J^{\circ}_{0,0}(t_0 \mid \delta_2 = 1) = f\big(pt^{\circ}_{2,1}(t_0) + t_0\big) + g\big(pt^{\circ}_{2,1}(t_0)\big)$$

illustrated in figure 64, is provided by lemma 2.

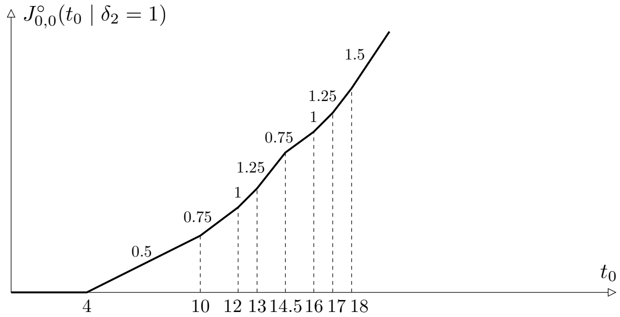


Figure 64: Conditioned cost-to-go $J^{\circ}_{0,0}(t_0 \mid \delta_2 = 1)$.

In order to find the optimal cost-to-go $J^{\circ}_{0,0}(t_0)$, it is necessary to carry out the following minimization

$$J^{\circ}_{0,0}(t_0) = \min\left\{J^{\circ}_{0,0}(t_0 \mid \delta_1 = 1)\,,\; J^{\circ}_{0,0}(t_0 \mid \delta_2 = 1)\right\}$$

which provides, in accordance with lemma 4, the function illustrated in figure 65.

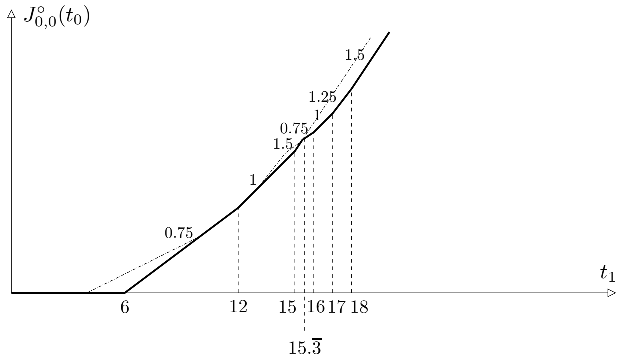


Figure 65: Optimal cost-to-go $J^{\circ}_{0,0}(t_0)$ in state $[0\ 0\ t_0]^T$.

Since $J^{\circ}_{0,0}(t_0 \mid \delta_1 = 1)$ is the minimum in $(-\infty, 15.\overline{3})$ and $J^{\circ}_{0,0}(t_0 \mid \delta_2 = 1)$ is the minimum in $[15.\overline{3}, +\infty)$, the optimal control strategies for this state are

$$\delta^{\circ}_1(0,0,t_0) = \begin{cases} 1 & t_0 < 15.\overline{3} \\ 0 & t_0 \geq 15.\overline{3} \end{cases} \qquad \delta^{\circ}_2(0,0,t_0) = \begin{cases} 0 & t_0 < 15.\overline{3} \\ 1 & t_0 \geq 15.\overline{3} \end{cases}$$

$$\tau^{\circ}(0,0,t_0) = \delta^{\circ}_1(0,0,t_0)\, pt^{\circ}_{1,1}(t_0) + \delta^{\circ}_2(0,0,t_0)\, pt^{\circ}_{2,1}(t_0) = \begin{cases} 4 & t_0 < 12 \\ -t_0 + 16 & 12 \leq t_0 < 15 \\ 2 & 15 \leq t_0 < 15.\overline{3} \\ 2 & 15.\overline{3} \leq t_0 < 16 \\ -t_0 + 18 & 16 \leq t_0 < 17 \\ 1 & t_0 \geq 17 \end{cases}$$

illustrated in figures 66, 67, and 68, respectively.

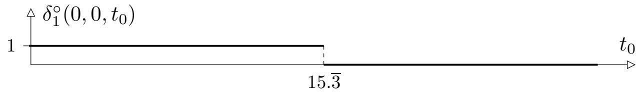


Figure 66: Optimal control strategy $\delta_1^\circ(0,0,t_0)$ in state $[0\ 0\ t_0]^T$.

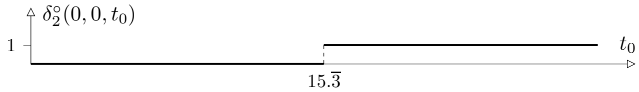


Figure 67: Optimal control strategy $\delta_2^\circ(0,0,t_0)$ in state $[0\ 0\ t_0]^T$.

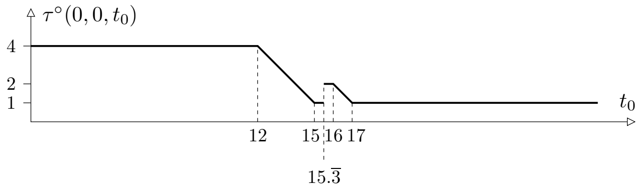


Figure 68: Optimal control strategy $\tau^\circ(0,0,t_0)$ (processing time) in state $[0\ 0\ t_0]^T$.

Since the two conditional costs-to-go have the same value in the interval $[10, 13]$, the following functions represent alternative optimal control strategies for the considered state

$$\delta_1^\circ(0,0,t_0) = \begin{cases} 1 & t_0 \le 10 \\ 0 & 10 \le t_0 < 13 \\ 1 & 13 \le t_0 < 15.\overline{3} \\ 0 & t_0 > 15.\overline{3} \end{cases} \qquad \delta_2^\circ(0,0,t_0) = \begin{cases} 0 & t_0 \le 10 \\ 1 & 10 \le t_0 < 13 \\ 0 & 13 \le t_0 < 15.\overline{3} \\ 1 & t_0 > 15.\overline{3} \end{cases}$$

$$\tau^\circ(0,0,t_1) = \delta_1^\circ(0,0,t_0)\, pt_{1,1}^\circ(t_0) + \delta_2^\circ(0,0,t_0)\, pt_{2,1}^\circ(t_0) = \begin{cases} 4 & t_0 < 10 \\ 2 & 10 \le t_0 < 12 \\ -t_0 + 14 & 12 \le t_0 < 13 \\ -t_0 + 16 & 13 \le t_0 < 15 \\ 1 & 15 \le t_0 < 15.\overline{3} \\ 2 & 15.\overline{3} \le t_0 < 16 \\ -t_0 + 18 & 16 \le t_0 < 17 \\ 1 & t_0 \ge 17 \end{cases}$$

Such functions are illustrated in figures 69, 70, and 71, respectively.

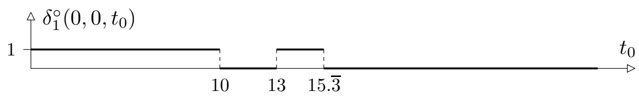


Figure 69: Alternative optimal control strategy $\delta_1^\circ(0,0,t_0)$ in state $[0\ 0\ t_0]^T$.

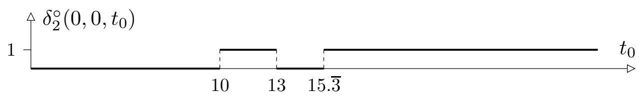


Figure 70: Alternative optimal control strategy $\delta_2^\circ(0,0,t_0)$ in state $[0\ 0\ t_0]^T$.

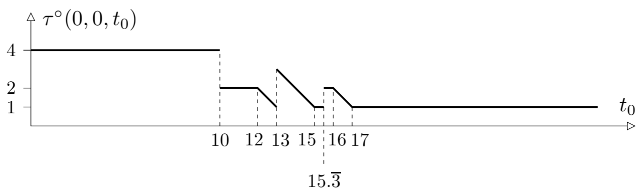


Figure 71: Alternative optimal control strategy $\tau^\circ(0,0,t_0)$ (service time) in state $[0\ 0\ t_0]^T$.

6 Application to the single machine scheduling – Example with setup

Consider a single machine scheduling problem in which 4 jobs of class P_1 and 3 jobs of class P_2 must be executed. The due dates, the marginal tardiness costs of jobs, the processing time bounds and the marginal deviation costs of jobs are:

$\alpha_{1,1} = 0.75$	$dd_{1,1} = 19$
$\alpha_{1,2} = 0.5$	$dd_{1,2} = 24$
$\alpha_{1,3} = 1.5$	$dd_{1,3} = 29$
$\alpha_{1,4} = 0.5$	$dd_{1,4} = 41$
$\beta_1 = 1$	
$pt_1^{\text{low}} = 4$	$pt_1^{\text{nom}} = 8$

$\alpha_{2,1} = 2$	$dd_{2,1} = 21$
$\alpha_{2,2} = 1$	$dd_{2,2} = 24$
$\alpha_{2,3} = 1$	$dd_{2,3} = 38$
$\beta_2 = 1.5$	
$pt_2^{\text{low}} = 4$	$pt_2^{\text{nom}} = 6$

A setup is required between the execution of jobs of different classes. Setup times and costs are:

$st_{1,1} = 0$	$st_{1,2} = 1$
$st_{2,1} = 0.5$	$st_{2,2} = 0$

$sc_{1,1} = 0$	$sc_{1,2} = 0.5$
$sc_{2,1} = 1$	$sc_{2,2} = 0$

The evolution of the system state can be represented by the following diagram.

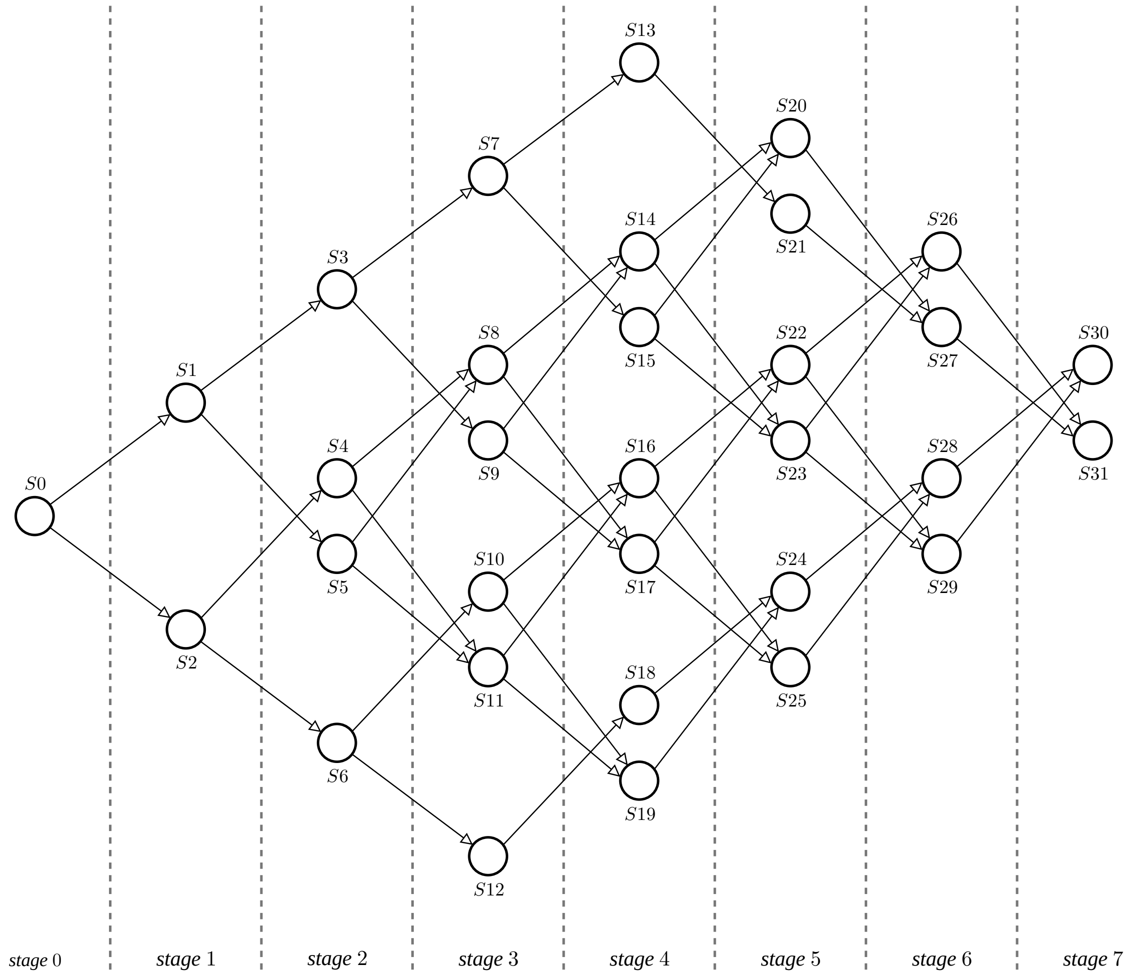


Figure 72: State diagram in the case of two classes of jobs, where $N_1 = 4$ and $N_2 = 3$, with setup.

The 32 states (from $S0$ to $S31$) in the 7 stages are:

stage 0	
$S0$	$[0\ 0\ 0\ t_0]^T$

stage 1	
$S1$	$[1\ 0\ 1\ t_1]^T$
$S2$	$[0\ 1\ 2\ t_1]^T$

stage 2	
$S3$	$[2\ 0\ 1\ t_2]^T$
$S4$	$[1\ 1\ 1\ t_2]^T$
$S5$	$[1\ 1\ 2\ t_2]^T$
$S6$	$[0\ 2\ 2\ t_2]^T$

stage 3	
$S7$	$[3\ 0\ 1\ t_3]^T$
$S8$	$[2\ 1\ 1\ t_3]^T$
$S9$	$[2\ 1\ 2\ t_3]^T$
$S10$	$[1\ 2\ 1\ t_3]^T$
$S11$	$[1\ 2\ 2\ t_3]^T$
$S12$	$[0\ 3\ 2\ t_3]^T$

stage 4	
$S13$	$[4\ 0\ 1\ t_4]^T$
$S14$	$[3\ 1\ 1\ t_4]^T$
$S15$	$[3\ 1\ 2\ t_4]^T$
$S16$	$[2\ 2\ 1\ t_4]^T$
$S17$	$[2\ 2\ 2\ t_4]^T$
$S18$	$[1\ 3\ 1\ t_4]^T$
$S19$	$[1\ 3\ 2\ t_4]^T$

stage 5	
$S20$	$[4\ 1\ 1\ t_5]^T$
$S21$	$[4\ 1\ 2\ t_5]^T$
$S22$	$[3\ 2\ 1\ t_5]^T$
$S23$	$[3\ 2\ 2\ t_5]^T$
$S24$	$[2\ 3\ 1\ t_5]^T$
$S25$	$[2\ 3\ 2\ t_5]^T$

stage 6	
$S26$	$[4\ 2\ 1\ t_6]^T$
$S27$	$[4\ 2\ 2\ t_6]^T$
$S28$	$[3\ 3\ 1\ t_6]^T$
$S29$	$[3\ 3\ 2\ t_6]^T$

stage 7	
$S30$	$[4\ 3\ 1\ t_7]^T$
$S31$	$[4\ 3\ 2\ t_7]^T$

The application of dynamic programming, in conjunction with the new lemmas, provides the following optimal control strategies.

Remark. In the following, the time variables t_j, $j = 0, \ldots, 7$, will be considered $\in \mathbb{R}$, that is, also negative values are taken into account. Negative values of t_j can be considered when the strategies are determined in advance with respect to the initial time instant 0 at which the processing of the jobs starts. In this case, it is possible to exploit the optimal control strategies determined for the negative values of t_j to start the execution of the jobs as soon as they become available, even before 0.

Stage 7 – State $[\mathbf{4\ 3\ 2}\ \boldsymbol{t_7}]^{\boldsymbol{T}}$ ($S31$)

No decision has to be taken in state $[4\ 3\ 2\ t_7]^T$. The optimal cost-to-go is obviously null, that is

$$J^\circ_{4,3,2}(t_7) = 0$$

Stage 7 – State $[\mathbf{4\ 3\ 1}\ \boldsymbol{t_7}]^{\boldsymbol{T}}$ ($S30$)

No decision has to be taken in state $[4\ 3\ 1\ t_7]^T$. The optimal cost-to-go is obviously null, that is

$$J^\circ_{4,3,1}(t_7) = 0$$

Stage 6 – State $[\mathbf{3\ 3\ 2}\ \boldsymbol{t_6}]^{\boldsymbol{T}}$ ($S29$)

In state $[3\ 3\ 2\ t_6]^T$ all jobs of class P_2 have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuos) decision variable τ only (which corresponds to the processing time $pt_{1,4}$), is

$$\alpha_{1,4}\ \max\{t_6 + st_{2,1} + \tau - dd_{1,4}\ ,\ 0\} + \beta_1\,(pt_1^{\text{nom}} - \tau) + sc_{2,1} + J^\circ_{4,3,1}(t_7)$$

that can be written as $f(pt_{1,4} + t_6) + g(pt_{1,4})$ being

$$f(pt_{1,4} + t_6) = 0.5 \cdot \max\{pt_{1,4} + t_6 - 40.5\ ,\ 0\} + 1$$

$$g(pt_{1,4}) = \begin{cases} 8 - pt_{1,4} & pt_{1,4} \in [4, 8) \\ 0 & pt_{1,4} \notin [4, 8) \end{cases}$$

The function $pt^\circ_{1,4}(t_6) = \arg\min_{pt_{1,4}}\{f(pt_{1,4} + t_6) + g(pt_{1,4})\}$, with $4 \le pt_{1,4} \le 8$, is determined by applying lemma 1. It is

$$pt^\circ_{1,4}(t_6) = x_{\text{e}}(t_6) \qquad \text{with} \quad x_{\text{e}}(t_6) = 8 \quad \forall\, t_6$$

Taking into account the mandatory decision about the class of the next job to be executed, the optimal control strategies for this state are

$$\delta^\circ_1(3,3,2,t_6) = 1 \quad \forall\, t_6 \qquad \delta^\circ_2(3,3,2,t_6) = 0 \quad \forall\, t_6$$

$\tau^{\circ}(3, 3, 2, t_6) = 8 \quad \forall\, t_6$

The optimal control strategy $\tau^{\circ}(3, 3, 2, t_6)$ is illustrated in figure 73.

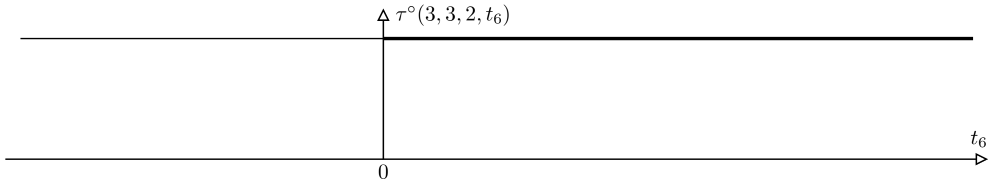


Figure 73: Optimal control strategy $\tau^{\circ}(3, 3, 2, t_6)$ in state $[3\ 3\ 2\ t_6]^T$.

The optimal cost-to-go $J^{\circ}_{3,3,2}(t_6) = f(pt^{\circ}_{1,4}(t_6) + t_6) + g(pt^{\circ}_{1,4}(t_6))$, illustrated in figure 74, is provided by lemma 2. It is specified by the initial value 1, by the abscissa $\gamma_1 = 32.5$ at which the slope changes, and by the slope $\mu_1 = 0.5$ in the interval $[32.5, +\infty)$.

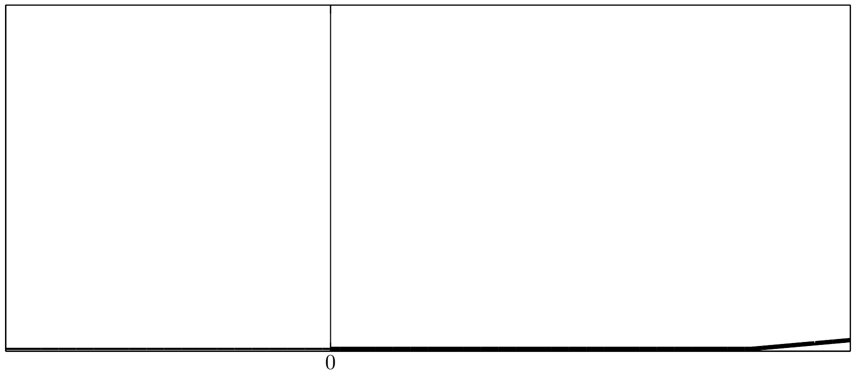


Figure 74: Optimal cost-to-go $J^{\circ}_{3,3,2}(t_6)$ in state $[3\ 3\ 2\ t_6]^T$.

Stage 6 – State $[3\ 3\ 1\ t_6]^T$ ($S28$)

In state $[3\ 3\ 1\ t_6]^T$ all jobs of class P_2 have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuos) decision variable τ only (which corresponds to the processing time $pt_{1,4}$), is

$$\alpha_{1,4}\, \max\{t_6 + st_{1,1} + \tau - dd_{1,4}\,,\, 0\} + \beta_1\, (pt_1^{\text{nom}} - \tau) + sc_{1,1} + J^{\circ}_{4,3,1}(t_7)$$

that can be written as $f(pt_{1,4} + t_6) + g(pt_{1,4})$ being

$$f(pt_{1,4} + t_6) = 0.5 \cdot \max\{pt_{1,4} + t_6 - 41\,,\, 0\}$$

$$g(pt_{1,4}) = \begin{cases} 8 - pt_{1,4} & pt_{1,4} \in [4, 8) \\ 0 & pt_{1,4} \notin [4, 8) \end{cases}$$

The function $pt^{\circ}_{1,4}(t_6) = \arg\min_{pt_{1,4}}\{f(pt_{1,4} + t_6) + g(pt_{1,4})\}$, with $4 \leq pt_{1,4} \leq 8$, is determined by applying lemma 1. It is

$$pt^{\circ}_{1,4}(t_6) = x_{\text{e}}(t_6) \qquad \text{with} \quad x_{\text{e}}(t_6) = 8 \quad \forall\, t_6$$

Taking into account the mandatory decision about the class of the next job to be executed, the optimal control strategies for this state are

$$\delta^{\circ}_1(3, 3, 1, t_6) = 1 \quad \forall\, t_6 \qquad \delta^{\circ}_2(3, 3, 1, t_6) = 0 \quad \forall\, t_6$$

$$\tau^{\circ}(3, 3, 1, t_6) = 8 \quad \forall\, t_6$$

The optimal control strategy $\tau^{\circ}(3, 3, 1, t_6)$ is illustrated in figure 75.

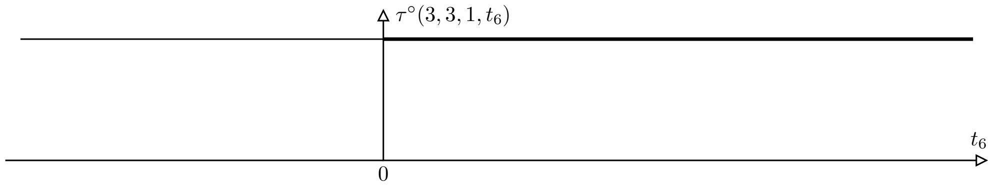


Figure 75: Optimal control strategy $\tau^\circ(3,3,1,t_6)$ in state $[3\ 3\ 1\ t_6]^T$.

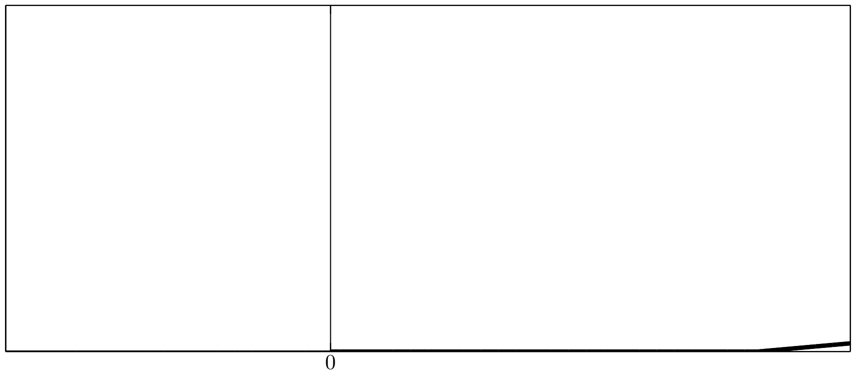


Figure 76: Optimal cost-to-go $J^\circ_{3,3,1}(t_6)$ in state $[3\ 3\ 1\ t_6]^T$.

The optimal cost-to-go $J^\circ_{3,3,1}(t_6) = f(pt^\circ_{1,4}(t_6) + t_6) + g(pt^\circ_{1,4}(t_6))$, illustrated in figure 76, is provided by lemma 2. It is specified by the initial value 0, by the abscissa $\gamma_1 = 33$ at which the slope changes, and by the slope $\mu_1 = 0.5$ in the interval $[33, +\infty)$.

Stage 6 – State $[4\ 2\ 2\ t_6]^T$ ($S27$)

In state $[4\ 2\ 2\ t_6]^T$ all jobs of class P_1 have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuos) decision variable τ only (which corresponds to the processing time $pt_{2,3}$), is

$$\alpha_{2,3}\ \max\{t_6 + st_{2,2} + \tau - dd_{2,3}\ ,\ 0\} + \beta_2\,(pt_2^{\text{nom}} - \tau) + sc_{2,2} + J^\circ_{4,3,2}(t_7)$$

that can be written as $f(pt_{2,3} + t_6) + g(pt_{2,3})$ being

$$f(pt_{2,3} + t_6) = \max\{pt_{2,3} + t_6 - 38\ ,\ 0\}$$

$$g(pt_{2,3}) = \begin{cases} 1.5 \cdot (6 - pt_{2,3}) & pt_{2,3} \in [4,6) \\ 0 & pt_{2,3} \notin [4,6) \end{cases}$$

The function $pt^\circ_{2,3}(t_6) = \arg\min_{pt_{2,3}}\{f(pt_{2,3} + t_6) + g(pt_{2,3})\}$, with $4 \le pt_{2,3} \le 6$, is determined by applying lemma 1. It is

$$pt^\circ_{2,3}(t_6) = x_{\text{e}}(t_6) \qquad \text{with} \quad x_{\text{e}}(t_6) = 6 \quad \forall\, t_6$$

Taking into account the mandatory decision about the class of the next job to be executed, the optimal control strategies for this state are

$$\delta^\circ_1(4,2,2,t_6) = 0 \quad \forall\, t_6 \qquad \delta^\circ_2(4,2,2,t_6) = 1 \quad \forall\, t_6$$

$$\tau^\circ(4,2,2,t_6) = 6 \quad \forall\, t_6$$

The optimal control strategy $\tau^\circ(4,2,2,t_6)$ is illustrated in figure 77.

The optimal cost-to-go $J^\circ_{4,2,1}(t_6) = f(pt^\circ_{2,3}(t_6) + t_6) + g(pt^\circ_{2,3}(t_6))$, illustrated in figure 78, is provided by lemma 2. It is specified by the initial value 0, by the abscissa $\gamma_1 = 32$ at which the slope changes, and by the slope $\mu_1 = 1$ in the interval $[32, +\infty)$.

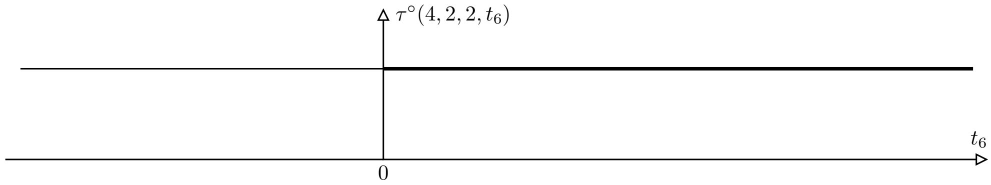


Figure 77: Optimal control strategy $\tau^\circ(4,2,2,t_6)$ in state $[4\ 2\ 2\ t_6]^T$.

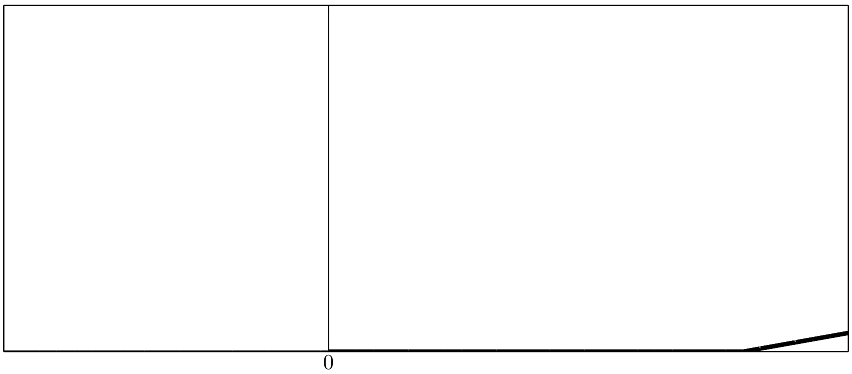


Figure 78: Optimal cost-to-go $J^\circ_{4,2,2}(t_6)$ in state $[4\ 2\ 2\ t_6]^T$.

Stage 6 – State $[\mathbf{4\ 2\ 1}\ \boldsymbol{t_6}]^{\boldsymbol{T}}$ ($S26$)

In state $[4\ 2\ 1\ t_6]^T$ all jobs of class P_1 have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuos) decision variable τ only (which corresponds to the processing time $pt_{2,3}$), is

$$\alpha_{2,3}\ \max\{t_6 + st_{1,2} + \tau - dd_{2,3}\ ,\ 0\} + \beta_2\,(pt_2^{\text{nom}} - \tau) + sc_{1,2} + J^\circ_{4,3,2}(t_7)$$

that can be written as $f(pt_{2,3} + t_6) + g(pt_{2,3})$ being

$$f(pt_{2,3} + t_6) = \max\{pt_{2,3} + t_6 - 37\ ,\ 0\} + 0.5$$

$$g(pt_{2,3}) = \begin{cases} 1.5 \cdot (6 - pt_{2,3}) & pt_{2,3} \in [4,6) \\ 0 & pt_{2,3} \notin [4,6) \end{cases}$$

The function $pt^\circ_{2,3}(t_6) = \arg\min_{pt_{2,3}}\{f(pt_{2,3} + t_6) + g(pt_{2,3})\}$, with $4 \le pt_{2,3} \le 6$, is determined by applying lemma 1. It is

$$pt^\circ_{2,3}(t_6) = x_{\rm e}(t_6) \qquad \text{with} \quad x_{\rm e}(t_6) = 6 \quad \forall\, t_6$$

Taking into account the mandatory decision about the class of the next job to be executed, the optimal control strategies for this state are

$$\delta^\circ_1(4,2,1,t_6) = 0 \quad \forall\, t_6 \qquad \delta^\circ_2(4,2,1,t_6) = 1 \quad \forall\, t_6$$

$$\tau^\circ(4,2,1,t_6) = 6 \quad \forall\, t_6$$

The optimal control strategy $\tau^\circ(4,2,1,t_6)$ is illustrated in figure 79.

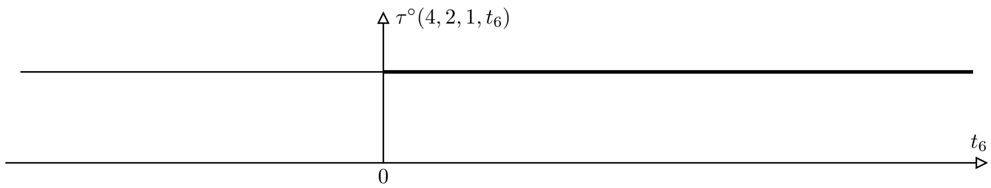


Figure 79: Optimal control strategy $\tau^\circ(4,2,1,t_6)$ in state $[4\ 2\ 1\ t_6]^T$.

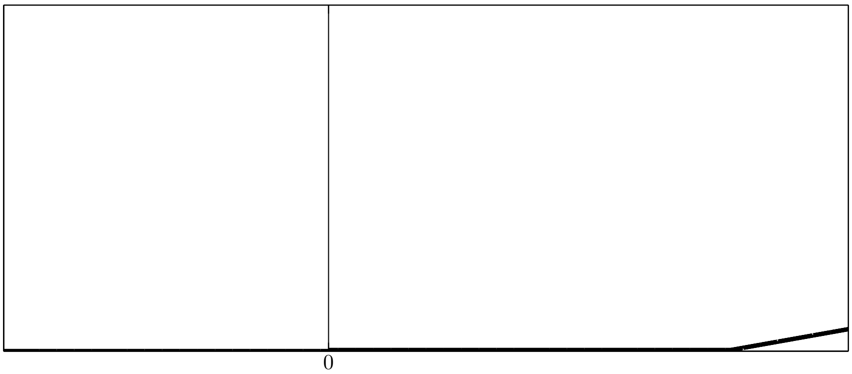


Figure 80: Optimal cost-to-go $J^{\circ}_{4,2,1}(t_6)$ in state $[4\ 2\ 1\ t_6]^T$.

The optimal cost-to-go $J^{\circ}_{4,2,1}(t_6) = f(pt^{\circ}_{2,3}(t_6) + t_6) + g(pt^{\circ}_{2,3}(t_6))$, illustrated in figure 80, is provided by lemma 2. It is specified by the initial value 0.5, by the abscissa $\gamma_1 = 31$ at which the slope changes, and by the slope $\mu_1 = 1$ in the interval $[31, +\infty)$.

Stage 5 – State $[2\ 3\ 2\ t_5]^T$ ($S25$)

In state $[2\ 3\ 2\ t_5]^T$ all jobs of class P_2 have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuos) decision variable τ only (which corresponds to the processing time $pt_{1,3}$), is

$$\alpha_{1,3}\ \max\{t_5 + st_{2,1} + \tau - dd_{1,3}\ ,\ 0\} + \beta_1\,(pt_1^{\mathrm{nom}} - \tau) + sc_{2,1} + J^{\circ}_{3,3,1}(t_6)$$

that can be written as $f(pt_{1,3} + t_5) + g(pt_{1,3})$ being

$$f(pt_{1,3} + t_5) = 1.5 \cdot \max\{pt_{1,3} + t_5 - 28.5\ ,\ 0\} + 1 + J^{\circ}_{3,3,1}(pt_{1,3} + t_5 + 0.5)$$

$$g(pt_{1,3}) = \begin{cases} 8 - pt_{1,3} & pt_{1,3} \in [4, 8) \\ 0 & pt_{1,3} \notin [4, 8) \end{cases}$$

The function $pt^{\circ}_{1,3}(t_5) = \arg\min_{pt_{1,3}}\{f(pt_{1,3} + t_5) + g(pt_{1,3})\}$, with $4 \leq pt_{1,3} \leq 8$, is determined by applying lemma 1. It is

$$pt^{\circ}_{1,3}(t_5) = x_{\mathrm{e}}(t_5) \qquad \text{with} \quad x_{\mathrm{e}}(t_5) = \begin{cases} 8 & t_5 < 20.5 \\ -t_5 + 28.5 & 20.5 \leq t_5 < 24.5 \\ 4 & t_5 \geq 24.5 \end{cases}$$

Taking into account the mandatory decision about the class of the next job to be executed, the optimal control strategies for this state are

$$\delta^{\circ}_1(2, 3, 2, t_5) = 1 \quad \forall\, t_5 \qquad \delta^{\circ}_2(2, 3, 2, t_5) = 0 \quad \forall\, t_5$$

$$\tau^{\circ}(2, 3, 2, t_5) = \begin{cases} 8 & t_5 < 20.5 \\ -t_5 + 28.5 & 20.5 \leq t_5 < 24.5 \\ 4 & t_5 \geq 24.5 \end{cases}$$

The optimal control strategy $\tau^{\circ}(2, 3, 2, t_5)$ is illustrated in figure 81.

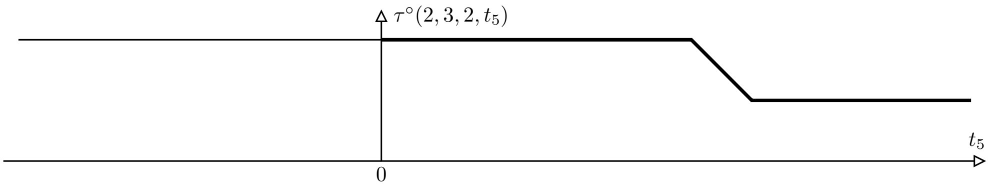


Figure 81: Optimal control strategy $\tau^{\circ}(2, 3, 2, t_5)$ in state $[2\ 3\ 2\ t_5]^T$.

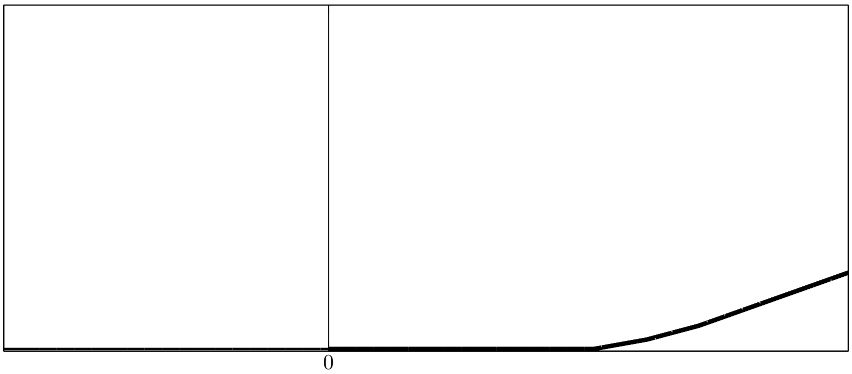


Figure 82: Optimal cost-to-go $J^\circ_{2,3,2}(t_5)$ in state $[2\ 3\ 2\ t_5]^T$.

The optimal cost-to-go $J^\circ_{2,3,2}(t_5) = f(pt^\circ_{1,3}(t_5) + t_5) + g(pt^\circ_{1,3}(t_5))$, illustrated in figure 82, is provided by lemma 2. It is specified by the initial value 1, by the set { 20.5, 24.5, 28.5 } of abscissae γ_i, $i = 1, \ldots, 3$, at which the slope changes, and by the set { 1, 1.5, 2 } of slopes μ_i, $i = 1, \ldots, 3$, in the various intervals.

Stage 5 – State $[\mathbf{2\ 3\ 1}\ \boldsymbol{t_5}]^{\boldsymbol{T}}$ ($S24$)

In state $[2\ 3\ 1\ t_5]^T$ all jobs of class P_2 have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuos) decision variable τ only (which corresponds to the processing time $pt_{1,3}$), is

$$\alpha_{1,3}\ \max\{t_5 + st_{1,1} + \tau - dd_{1,3}\ , \ 0\} + \beta_1\ (pt_1^{\text{nom}} - \tau) + sc_{1,1} + J^\circ_{3,3,1}(t_6)$$

that can be written as $f(pt_{1,3} + t_5) + g(pt_{1,3})$ being

$$f(pt_{1,3} + t_5) = 1.5 \cdot \max\{pt_{1,3} + t_5 - 29\ , \ 0\} + J^\circ_{3,3,1}(pt_{1,3} + t_5)$$

$$g(pt_{1,3}) = \begin{cases} 8 - pt_{1,3} & pt_{1,3} \in [4, 8) \\ 0 & pt_{1,3} \notin [4, 8) \end{cases}$$

The function $pt^\circ_{1,3}(t_5) = \arg\min_{pt_{1,3}}\{f(pt_{1,3} + t_5) + g(pt_{1,3})\}$, with $4 \leq pt_{1,3} \leq 8$, is determined by applying lemma 1. It is

$$pt^\circ_{1,3}(t_5) = x_\text{e}(t_5) \qquad \text{with} \quad x_\text{e}(t_5) = \begin{cases} 8 & t_5 < 21 \\ -t_5 + 29 & 21 \leq t_5 < 25 \\ 4 & t_5 \geq 25 \end{cases}$$

Taking into account the mandatory decision about the class of the next job to be executed, the optimal control strategies for this state are

$$\delta^\circ_1(2, 3, 1, t_5) = 1 \quad \forall\, t_5 \qquad \delta^\circ_2(2, 3, 1, t_5) = 0 \quad \forall\, t_5$$

$$\tau^\circ(2, 3, 1, t_5) = \begin{cases} 8 & t_5 < 21 \\ -t_5 + 29 & 21 \leq t_5 < 25 \\ 4 & t_5 \geq 25 \end{cases}$$

The optimal control strategy $\tau^\circ(2, 3, 1, t_5)$ is illustrated in figure 83.

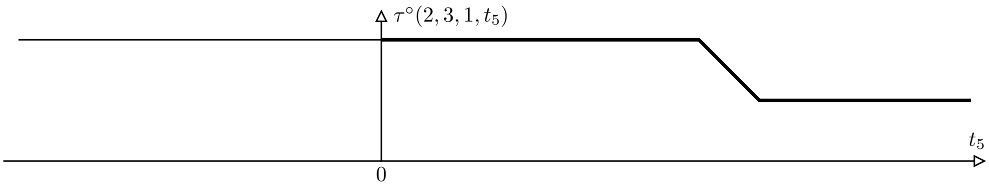


Figure 83: Optimal control strategy $\tau^\circ(2, 3, 1, t_5)$ in state $[2\ 3\ 1\ t_5]^T$.

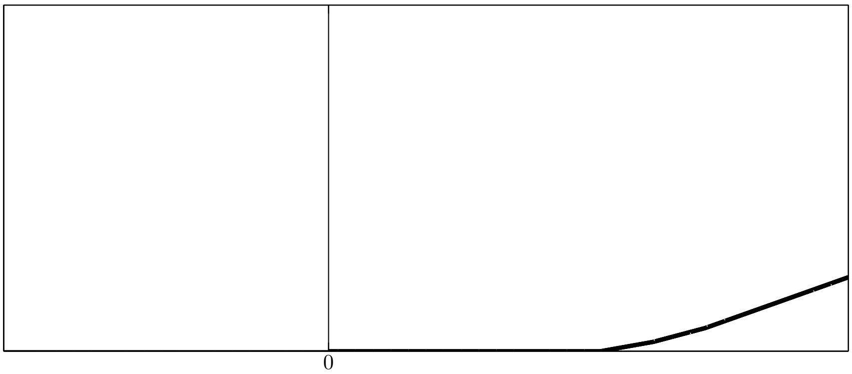


Figure 84: Optimal cost-to-go $J^{\circ}_{2,3,1}(t_5)$ in state $[2\ 3\ 1\ t_5]^T$.

The optimal cost-to-go $J^{\circ}_{2,3,1}(t_5) = f(pt^{\circ}_{1,3}(t_5) + t_5) + g(pt^{\circ}_{1,3}(t_5))$, illustrated in figure 84, is provided by lemma 2. It is specified by the initial value 0, by the set { 21, 25, 29 } of abscissae γ_i, $i = 1, \ldots, 3$, at which the slope changes, and by the set { 1, 1.5, 2 } of slopes μ_i, $i = 1, \ldots, 3$, in the various intervals.

Stage 5 – State $[3\ 2\ 2\ t_5]^T$ ($S23$)

In state $[3\ 2\ 2\ t_5]^T$, the cost function to be minimized, with respect to the (continuos) decision variable τ and to the (binary) decision variables δ_1 and δ_2 is

$$\delta_1\big[\alpha_{1,4}\ \max\{t_5 + st_{2,1} + \tau - dd_{1,4}\ ,\ 0\} + \beta_1\,(pt_1^{\text{nom}} - \tau) + sc_{2,1} + J^{\circ}_{4,2,1}(t_6)\big] +$$
$$+\ \delta_2\big[\alpha_{2,3}\ \max\{t_5 + st_{2,2} + \tau - dd_{2,3}\ ,\ 0\} + \beta_2\,(pt_2^{\text{nom}} - \tau) + sc_{2,2} + J^{\circ}_{3,3,2}(t_6)\big]$$

Case i) in which it is assumed $\delta_1 = 1$ (and $\delta_2 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{1,4}$, the following function

$$\alpha_{1,4}\ \max\{t_5 + st_{2,1} + \tau - dd_{1,4}\ ,\ 0\} + \beta_1\,(pt_1^{\text{nom}} - \tau) + sc_{2,1} + J^{\circ}_{4,2,1}(t_6)$$

that can be written as $f(pt_{1,4} + t_5) + g(pt_{1,4})$ being

$$f(pt_{1,4} + t_5) = 0.5 \cdot \max\{pt_{1,4} + t_5 - 40.5\ ,\ 0\} + 1 + J^{\circ}_{4,2,1}(pt_{1,4} + t_5 + 0.5)$$

$$g(pt_{1,4}) = \begin{cases} 8 - pt_{1,4} & pt_{1,4} \in [4, 8) \\ 0 & pt_{1,4} \notin [4, 8) \end{cases}$$

The function $pt^{\circ}_{1,4}(t_5) = \arg\min_{pt_{1,4}}\{f(pt_{1,4} + t_5) + g(pt_{1,4})\}$, with $4 \leq pt_{1,4} \leq 8$, is determined by applying lemma 1. It is (see figure 85)

$$pt^{\circ}_{1,4}(t_5) = x_{\text{e}}(t_5) \qquad \text{with} \quad x_{\text{e}}(t_5) = \begin{cases} 8 & t_5 < 22.5 \\ -t_5 + 30.5 & 22.5 \leq t_5 < 26.5 \\ 4 & t_5 \geq 26.5 \end{cases}$$

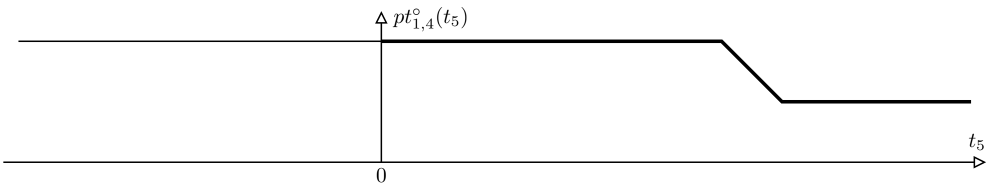


Figure 85: Optimal processing time $pt^{\circ}_{1,4}(t_5)$, under the assumption $\delta_1 = 1$ in state $[3\ 2\ 2\ t_5]^T$.

The conditioned cost-to-go $J^{\circ}_{3,2,2}(t_5 \mid \delta_1 = 1) = f(pt^{\circ}_{1,4}(t_5) + t_5) + g(pt^{\circ}_{1,4}(t_5))$, illustrated in figure 87, is provided by lemma 2. It is specified by the initial value 1.5, by the set { 22.5, 36.5 } of abscissae γ_i, $i = 1, \ldots, 2$, at which the slope changes, and by the set { 1, 1.5 } of slopes μ_i, $i = 1, \ldots, 2$, in the various intervals.

Case ii) in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{2,3}$, the following function

$$\alpha_{2,3}\ \max\{t_5 + st_{2,2} + \tau - dd_{2,3}\ ,\ 0\} + \beta_2\,(pt_2^{\mathrm{nom}} - \tau) + sc_{2,2} + J^{\circ}_{3,3,2}(t_6)$$

that can be written as $f(pt_{2,3} + t_5) + g(pt_{2,3})$ being

$$f(pt_{2,3} + t_5) = \max\{pt_{2,3} + t_5 - 38\ ,\ 0\} + J^{\circ}_{3,3,2}(pt_{2,3} + t_5)$$

$$g(pt_{2,3}) = \begin{cases} 1.5 \cdot (6 - pt_{2,3}) & pt_{2,3} \in [4,6) \\ 0 & pt_{2,3} \notin [4,6) \end{cases}$$

The function $pt^{\circ}_{2,3}(t_5) = \arg\min_{pt_{2,3}}\{f(pt_{2,3} + t_5) + g(pt_{2,3})\}$, with $4 \leq pt_{2,3} \leq 6$, is determined by applying lemma 1. It is (see figure 86)

$$pt^{\circ}_{2,3}(t_5) = x_{\mathrm{e}}(t_5) \qquad \text{with} \quad x_{\mathrm{e}}(t_5) = \begin{cases} 6 & t_5 < 32 \\ -t_5 + 38 & 32 \leq t_5 < 34 \\ 4 & t_5 \geq 34 \end{cases}$$

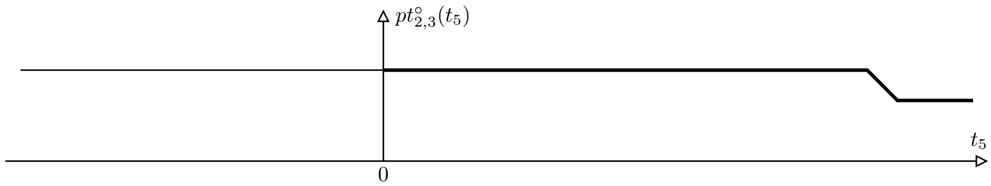


Figure 86: Optimal processing time $pt^{\circ}_{2,3}(t_5)$, under the assumption $\delta_2 = 1$ in state $[3\ 2\ 2\ t_5]^T$.

The conditioned cost-to-go $J^{\circ}_{3,2,2}(t_5 \mid \delta_2 = 1) = f(pt^{\circ}_{2,3}(t_5) + t_5) + g(pt^{\circ}_{2,3}(t_5))$, illustrated in figure 87, is provided by lemma 2. It is specified by the initial value 1, by the set { 26.5, 32 } of abscissae γ_i, $i = 1, \ldots, 2$, at which the slope changes, and by the set { 0.5, 1.5 } of slopes μ_i, $i = 1, \ldots, 2$, in the various intervals.

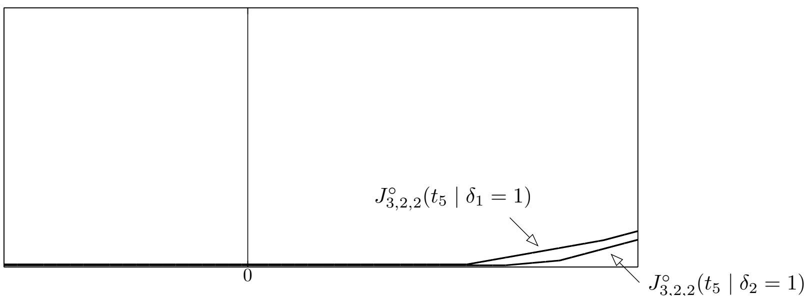


Figure 87: Conditioned costs-to-go $J^{\circ}_{3,2,2}(t_5 \mid \delta_1 = 1)$ and $J^{\circ}_{3,2,2}(t_5 \mid \delta_2 = 1)$ in state $[3\ 2\ 2\ t_5]^T$.

In order to find the optimal cost-to-go $J^{\circ}_{3,2,2}(t_5)$, it is necessary to carry out the following minimization

$$J^{\circ}_{3,2,2}(t_5) = \min\left\{J^{\circ}_{3,2,2}(t_5 \mid \delta_1 = 1)\,,\, J^{\circ}_{3,2,2}(t_5 \mid \delta_2 = 1)\right\}$$

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 88.

The function $J^{\circ}_{3,2,2}(t_5)$ is specified by the initial value 1, by the set { 26.5, 32 } of abscissae γ_i, $i = 1, \ldots, 2$, at which the slope changes, and by the set { 0.5, 1.5 } of slopes μ_i, $i = 1, \ldots, 2$, in the various intervals.

Since $J^{\circ}_{3,2,2}(t_5 \mid \delta_2 = 1)$ is always the minimum (see again figure 87), the optimal control strategies for this state are

$$\delta^{\circ}_1(3,2,2,t_5) = 0 \quad \forall\, t_5 \qquad \delta^{\circ}_2(3,2,2,t_5) = 1 \quad \forall\, t_5$$

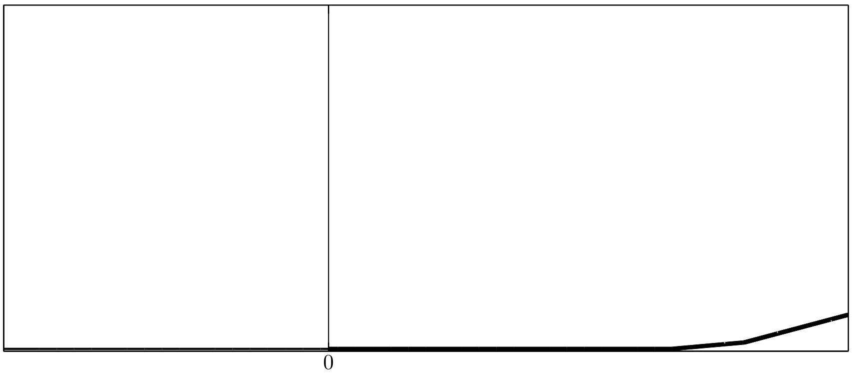


Figure 88: Optimal cost-to-go $J^\circ_{3,2,2}(t_5)$ in state $[3\ 2\ 2\ t_5]^T$.

$$\tau^\circ(3,2,2,t_5) = \begin{cases} 6 & t_5 < 32 \\ -t_5 + 38 & 32 \le t_5 < 34 \\ 4 & t_5 \ge 34 \end{cases}$$

The optimal control strategy $\tau^\circ(3,2,2,t_5)$ is illustrated in figure 89.

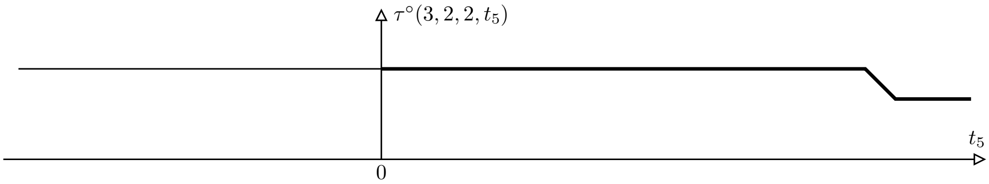


Figure 89: Optimal control strategy $\tau^\circ(3,2,2,t_5)$ in state $[3\ 2\ 2\ t_5]^T$.

Stage 5 – State $[\mathbf{3\ 2\ 1}\ \boldsymbol{t_5}]^{\boldsymbol{T}}$ ($S22$)

In state $[3\ 2\ 1\ t_5]^T$, the cost function to be minimized, with respect to the (continuos) decision variable τ and to the (binary) decision variables δ_1 and δ_2 is

$$\delta_1 \big[\alpha_{1,4}\ \max\{t_5 + st_{1,1} + \tau - dd_{1,4}\ , 0\} + \beta_1\,(pt_1^{\text{nom}} - \tau) + sc_{1,1} + J^\circ_{4,2,1}(t_6)\big] +$$
$$+\ \delta_2 \big[\alpha_{2,3}\ \max\{t_5 + st_{1,2} + \tau - dd_{2,3}\ , 0\} + \beta_2\,(pt_2^{\text{nom}} - \tau) + sc_{1,2} + J^\circ_{3,3,2}(t_6)\big]$$

Case i) in which it is assumed $\delta_1 = 1$ (and $\delta_2 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{1,4}$, the following function

$$\alpha_{1,4}\ \max\{t_5 + st_{1,1} + \tau - dd_{1,4}\ , 0\} + \beta_1\,(pt_1^{\text{nom}} - \tau) + sc_{1,1} + J^\circ_{4,2,1}(t_6)$$

that can be written as $f(pt_{1,4} + t_5) + g(pt_{1,4})$ being

$$f(pt_{1,4} + t_5) = 0.5 \cdot \max\{pt_{1,4} + t_5 - 41\ , 0\} + J^\circ_{4,2,1}(pt_{1,4} + t_5)$$

$$g(pt_{1,4}) = \begin{cases} 8 - pt_{1,4} & pt_{1,4} \in [4,8) \\ 0 & pt_{1,4} \notin [4,8) \end{cases}$$

The function $pt^\circ_{1,4}(t_5) = \arg\min_{pt_{1,4}}\{f(pt_{1,4} + t_5) + g(pt_{1,4})\}$, with $4 \le pt_{1,4} \le 8$, is determined by applying lemma 1. It is (see figure 90)

$$pt^\circ_{1,4}(t_5) = x_{\text{e}}(t_5) \qquad \text{with} \quad x_{\text{e}}(t_5) = \begin{cases} 8 & t_5 < 23 \\ -t_5 + 31 & 23 \le t_5 < 27 \\ 4 & t_5 \ge 27 \end{cases}$$

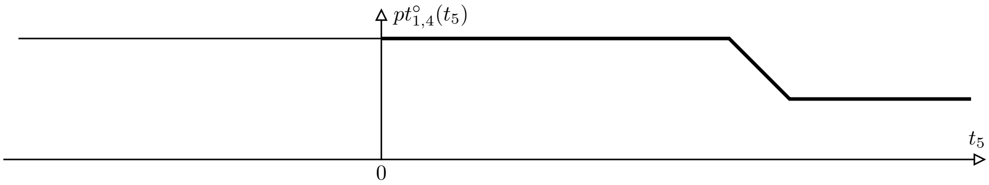


Figure 90: Optimal processing time $pt^{\circ}_{1,4}(t_5)$, under the assumption $\delta_1 = 1$ in state $[3\ 2\ 1\ t_5]^T$.

The conditioned cost-to-go $J^{\circ}_{3,2,1}(t_5 \mid \delta_1 = 1) = f(pt^{\circ}_{1,4}(t_5) + t_5) + g(pt^{\circ}_{1,4}(t_5))$, illustrated in figure 92, is provided by lemma 2. It is specified by the initial value 0.5, by the set { 23, 37 } of abscissae γ_i, $i = 1, \ldots, 2$, at which the slope changes, and by the set { 1, 1.5 } of slopes μ_i, $i = 1, \ldots, 2$, in the various intervals.

Case ii) in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{2,3}$, the following function

$$\alpha_{2,3}\ \max\{t_5 + st_{1,2} + \tau - dd_{2,3}\ ,\ 0\} + \beta_2\,(pt_2^{\mathrm{nom}} - \tau) + sc_{1,2} + J^{\circ}_{3,3,2}(t_6)$$

that can be written as $f(pt_{2,3} + t_5) + g(pt_{2,3})$ being

$$f(pt_{2,3} + t_5) = \max\{pt_{2,3} + t_5 - 37\ ,\ 0\} + 0.5 + J^{\circ}_{3,3,2}(pt_{2,3} + t_5 + 1)$$

$$g(pt_{2,3}) = \begin{cases} 1.5 \cdot (6 - pt_{2,3}) & pt_{2,3} \in [4, 6) \\ 0 & pt_{2,3} \notin [4, 6) \end{cases}$$

The function $pt^{\circ}_{2,3}(t_5) = \arg\min_{pt_{2,3}}\{f(pt_{2,3} + t_5) + g(pt_{2,3})\}$, with $4 \le pt_{2,3} \le 6$, is determined by applying lemma 1. It is (see figure 91)

$$pt^{\circ}_{2,3}(t_5) = x_{\mathrm{e}}(t_5) \qquad \text{with} \quad x_{\mathrm{e}}(t_5) = \begin{cases} 6 & t_5 < 31 \\ -t_5 + 37 & 31 \le t_5 < 33 \\ 4 & t_5 \ge 33 \end{cases}$$

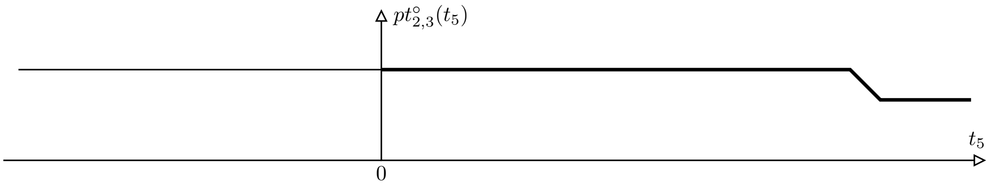


Figure 91: Optimal processing time $pt^{\circ}_{2,3}(t_5)$, under the assumption $\delta_2 = 1$ in state $[3\ 2\ 1\ t_5]^T$.

The conditioned cost-to-go $J^{\circ}_{3,2,1}(t_5 \mid \delta_2 = 1) = f(pt^{\circ}_{2,3}(t_5) + t_5) + g(pt^{\circ}_{2,3}(t_5))$, illustrated in figure 92, is provided by lemma 2. It is specified by the initial value 1.5, by the set { 25.5, 31 } of abscissae γ_i, $i = 1, \ldots, 2$, at which the slope changes, and by the set { 0.5, 1.5 } of slopes μ_i, $i = 1, \ldots, 2$, in the various intervals.

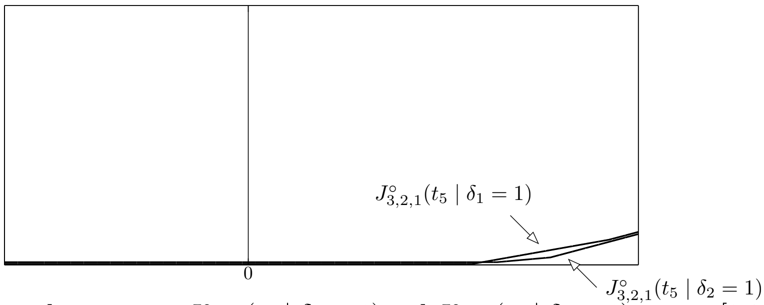


Figure 92: Conditioned costs-to-go $J^{\circ}_{3,2,1}(t_5 \mid \delta_1 = 1)$ and $J^{\circ}_{3,2,1}(t_5 \mid \delta_2 = 1)$ in state $[3\ 2\ 1\ t_5]^T$.

In order to find the optimal cost-to-go $J^{\circ}_{3,2,1}(t_5)$, it is necessary to carry out the following minimization

$$J^{\circ}_{3,2,1}(t_5) = \min\left\{J^{\circ}_{3,2,1}(t_5 \mid \delta_1 = 1)\,,\; J^{\circ}_{3,2,1}(t_5 \mid \delta_2 = 1)\right\}$$

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 93.

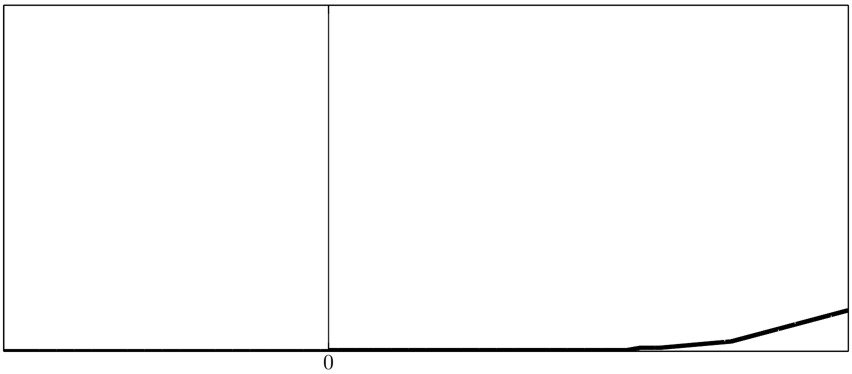


Figure 93: Optimal cost-to-go $J^{\circ}_{3,2,1}(t_5)$ in state $[3\ 2\ 1\ t_5]^T$.

The function $J^{\circ}_{3,2,1}(t_5)$ is specified by the initial value 0.5, by the set { 23, 24, 25.5, 31 } of abscissae γ_i, $i = 1, \ldots, 4$, at which the slope changes, and by the set { 1, 0, 0.5, 1.5 } of slopes μ_i, $i = 1, \ldots, 4$, in the various intervals.

Since $J^{\circ}_{3,2,1}(t_5 \mid \delta_1 = 1)$ is the minimum in $(-\infty, 24)$, and $J^{\circ}_{3,2,1}(t_5 \mid \delta_2 = 1)$ is the minimum in $[24, +\infty)$, the optimal control strategies for this state are

$$\delta^{\circ}_1(3,2,1,t_5) = \begin{cases} 1 & t_5 < 24 \\ 0 & t_5 \geq 24 \end{cases} \qquad \delta^{\circ}_2(3,2,1,t_5) = \begin{cases} 0 & t_5 < 24 \\ 1 & t_5 \geq 24 \end{cases}$$

$$\tau^{\circ}(3,2,1,t_5) = \begin{cases} 8 & t_5 < 23 \\ -t_5 + 31 & 23 \leq t_5 < 24 \\ 6 & 24 \leq t_5 < 31 \\ -t_5 + 37 & 31 \leq t_5 < 33 \\ 4 & t_5 \geq 33 \end{cases}$$

The optimal control strategy $\tau^{\circ}(3,2,1,t_5)$ is illustrated in figure 94.

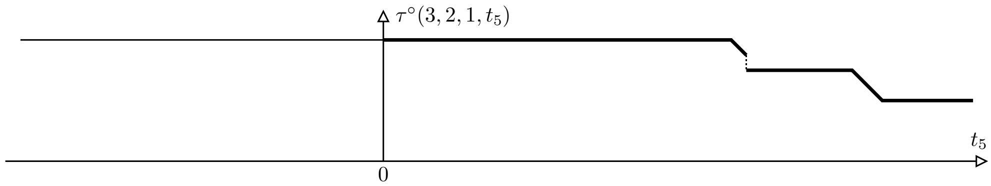


Figure 94: Optimal control strategy $\tau^{\circ}(3,2,1,t_5)$ in state $[3\ 2\ 1\ t_5]^T$.

Stage 5 – State $[4\ 1\ 2\ t_5]^T$ ($S21$)

In state $[4\ 1\ 2\ t_5]^T$ all jobs of class P_1 have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuos) decision variable τ only (which corresponds to the processing time $pt_{2,2}$), is

$$\alpha_{2,2}\, \max\{t_5 + st_{2,2} + \tau - dd_{2,2}\,,\, 0\} + \beta_2\, (pt_2^{\text{nom}} - \tau) + sc_{2,2} + J^{\circ}_{4,2,2}(t_6)$$

that can be written as $f(pt_{2,2} + t_5) + g(pt_{2,2})$ being

$$f(pt_{2,2} + t_5) = \max\{pt_{2,2} + t_5 - 24\,,\, 0\} + J^{\circ}_{4,2,2}(pt_{2,2} + t_5)$$

$$g(pt_{2,2}) = \begin{cases} 1.5 \cdot (6 - pt_{2,2}) & pt_{2,2} \in [4,6) \\ 0 & pt_{2,2} \notin [4,6) \end{cases}$$

The function $pt^{\circ}_{2,2}(t_5) = \arg\min_{pt_{2,2}}\{f(pt_{2,2} + t_5) + g(pt_{2,2})\}$, with $4 \leq pt_{2,2} \leq 6$, is determined by applying lemma 1. It is

$$pt^{\circ}_{2,2}(t_5) = x_{\mathrm{e}}(t_5) \qquad \text{with} \quad x_{\mathrm{e}}(t_5) = \begin{cases} 6 & t_5 < 26 \\ -t_5 + 32 & 26 \leq t_5 < 28 \\ 4 & t_5 \geq 28 \end{cases}$$

Taking into account the mandatory decision about the class of the next job to be executed, the optimal control strategies for this state are

$$\delta^{\circ}_1(4,1,2,t_5) = 0 \quad \forall\, t_5 \qquad \delta^{\circ}_2(4,1,2,t_5) = 1 \quad \forall\, t_5$$

$$\tau^{\circ}(4,1,2,t_5) = \begin{cases} 6 & t_5 < 26 \\ -t_5 + 32 & 26 \leq t_5 < 28 \\ 4 & t_5 \geq 28 \end{cases}$$

The optimal control strategy $\tau^{\circ}(4,1,2,t_5)$ is illustrated in figure 95.

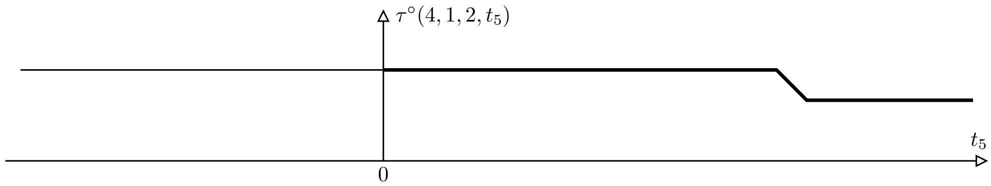


Figure 95: Optimal control strategy $\tau^{\circ}(4,1,2,t_5)$ in state $[4\ 1\ 2\ t_5]^T$.

The optimal cost-to-go $J^{\circ}_{4,1,2}(t_5) = f(pt^{\circ}_{2,2}(t_5) + t_5) + g(pt^{\circ}_{2,2}(t_5))$, illustrated in figure 96, is provided by lemma 2. It is specified by the initial value 0, by the set { 18, 26, 28 } of abscissae γ_i, $i = 1, \ldots, 3$, at which the slope changes, and by the set { 1, 1.5, 2 } of slopes μ_i, $i = 1, \ldots, 3$, in the various intervals.

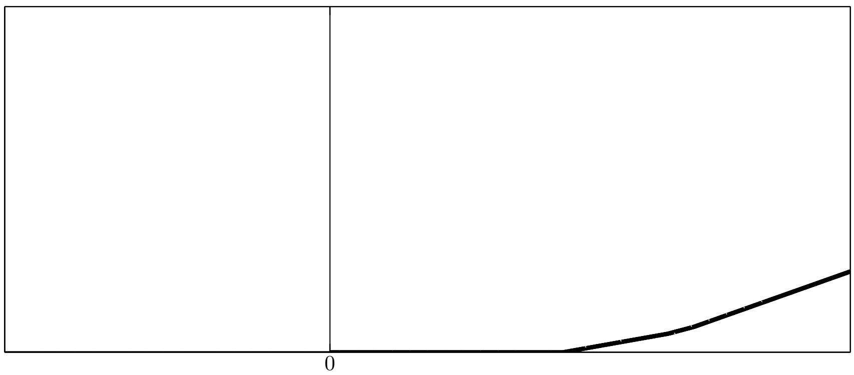


Figure 96: Optimal cost-to-go $J^{\circ}_{4,1,2}(t_5)$ in state $[4\ 1\ 2\ t_5]^T$.

Stage 5 – State $[4\ 1\ 1\ t_5]^T$ ($S20$)

In state $[4\ 1\ 1\ t_5]^T$ all jobs of class P_1 have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuos) decision variable τ only (which corresponds to the processing time $pt_{2,2}$), is

$$\alpha_{2,2}\ \max\{t_5 + st_{1,2} + \tau - dd_{2,2}\ ,\ 0\} + \beta_2\,(pt^{\mathrm{nom}}_2 - \tau) + sc_{1,2} + J^{\circ}_{4,2,2}(t_6)$$

that can be written as $f(pt_{2,2} + t_5) + g(pt_{2,2})$ being

$$f(pt_{2,2} + t_5) = \max\{pt_{2,2} + t_5 - 23\ ,\ 0\} + 0.5 + J^{\circ}_{4,2,2}(pt_{2,2} + t_5 + 1)$$

$$g(pt_{2,2}) = \begin{cases} 1.5 \cdot (6 - pt_{2,2}) & pt_{2,2} \in [4, 6) \\ 0 & pt_{2,2} \notin [4, 6) \end{cases}$$

The function $pt^{\circ}_{2,2}(t_5) = \arg\min_{pt_{2,2}}\{f(pt_{2,2} + t_5) + g(pt_{2,2})\}$, with $4 \leq pt_{2,2} \leq 6$, is determined by applying lemma 1. It is

$$pt^{\circ}_{2,2}(t_5) = x_{\mathrm{e}}(t_5) \qquad \text{with} \quad x_{\mathrm{e}}(t_5) = \begin{cases} 6 & t_5 < 25 \\ -t_5 + 31 & 25 \leq t_5 < 27 \\ 4 & t_5 \geq 27 \end{cases}$$

Taking into account the mandatory decision about the class of the next job to be executed, the optimal control strategies for this state are

$$\delta^{\circ}_1(4, 1, 1, t_5) = 0 \quad \forall\, t_5 \qquad \delta^{\circ}_2(4, 1, 1, t_5) = 1 \quad \forall\, t_5$$

$$\tau^{\circ}(4, 1, 1, t_5) = \begin{cases} 6 & t_5 < 25 \\ -t_5 + 31 & 25 \leq t_5 < 27 \\ 4 & t_5 \geq 27 \end{cases}$$

The optimal control strategy $\tau^{\circ}(4, 1, 1, t_5)$ is illustrated in figure 97.

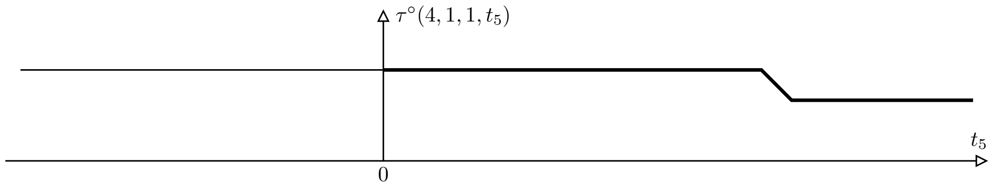


Figure 97: Optimal control strategy $\tau^{\circ}(4, 1, 1, t_5)$ in state $[4\ 1\ 1\ t_5]^T$.

The optimal cost-to-go $J^{\circ}_{4,1,1}(t_5) = f(pt^{\circ}_{2,2}(t_5) + t_5) + g(pt^{\circ}_{2,2}(t_5))$, illustrated in figure 98, is provided by lemma 2. It is specified by the initial value 0.5, by the set { 17, 25, 27 } of abscissae γ_i, $i = 1, \ldots, 3$, at which the slope changes, and by the set { 1, 1.5, 2 } of slopes μ_i, $i = 1, \ldots, 3$, in the various intervals.

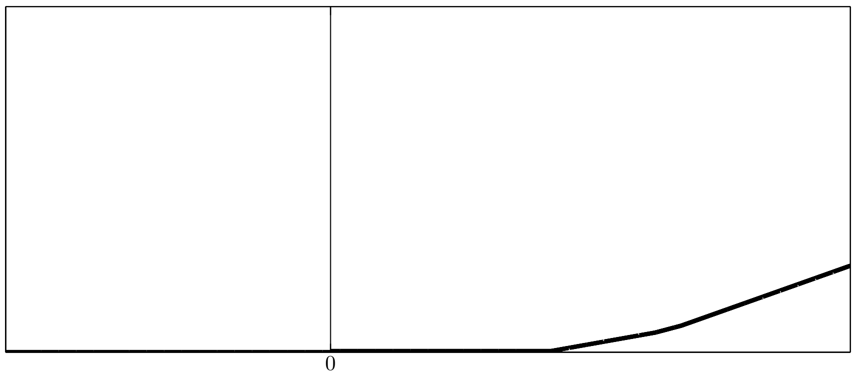


Figure 98: Optimal cost-to-go $J^{\circ}_{4,1,1}(t_5)$ in state $[4\ 1\ 1\ t_5]^T$.

Stage 4 – State $[\mathbf{1\ 3\ 2}\ \boldsymbol{t_4}]^{\boldsymbol{T}}$ ($S19$)

In state $[1\ 3\ 2\ t_4]^T$ all jobs of class P_2 have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuos) decision variable τ only (which corresponds to the processing time $pt_{1,2}$), is

$$\alpha_{1,2}\ \max\{t_4 + st_{2,1} + \tau - dd_{1,2}\ ,\ 0\} + \beta_1\,(pt^{\mathrm{nom}}_1 - \tau) + sc_{2,1} + J^{\circ}_{2,3,1}(t_5)$$

that can be written as $f(pt_{1,2} + t_4) + g(pt_{1,2})$ being

$$f(pt_{1,2} + t_4) = 0.5 \cdot \max\{pt_{1,2} + t_4 - 23.5\ ,\ 0\} + 1 + J^{\circ}_{2,3,1}(pt_{1,2} + t_4 + 0.5)$$

$$g(pt_{1,2}) = \begin{cases} 8 - pt_{1,2} & pt_{1,2} \in [4, 8) \\ 0 & pt_{1,2} \notin [4, 8) \end{cases}$$

The function $pt^{\circ}_{1,2}(t_4) = \arg\min_{pt_{1,2}}\{f(pt_{1,2} + t_4) + g(pt_{1,2})\}$, with $4 \leq pt_{1,2} \leq 8$, is determined by applying lemma 1. It is

$$pt^{\circ}_{1,2}(t_4) = x_{\mathrm{e}}(t_4) \qquad \text{with} \quad x_{\mathrm{e}}(t_4) = \begin{cases} 8 & t_4 < 12.5 \\ -t_4 + 20.5 & 12.5 \leq t_4 < 16.5 \\ 4 & t_4 \geq 16.5 \end{cases}$$

Taking into account the mandatory decision about the class of the next job to be executed, the optimal control strategies for this state are

$$\delta^{\circ}_1(1, 3, 2, t_4) = 1 \quad \forall\, t_4 \qquad \delta^{\circ}_2(1, 3, 2, t_4) = 0 \quad \forall\, t_4$$

$$\tau^{\circ}(1, 3, 2, t_4) = \begin{cases} 8 & t_4 < 12.5 \\ -t_4 + 20.5 & 12.5 \leq t_4 < 16.5 \\ 4 & t_4 \geq 16.5 \end{cases}$$

The optimal control strategy $\tau^{\circ}(1, 3, 2, t_4)$ is illustrated in figure 99.

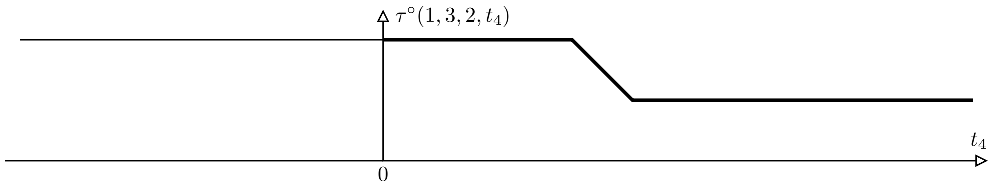


Figure 99: Optimal control strategy $\tau^{\circ}(1, 3, 2, t_4)$ in state $[1\ 3\ 2\ t_4]^T$.

The optimal cost-to-go $J^{\circ}_{1,3,2}(t_4) = f(pt^{\circ}_{1,2}(t_4) + t_4) + g(pt^{\circ}_{1,2}(t_4))$, illustrated in figure 100, is provided by lemma 2. It is specified by the initial value 1, by the set { 12.5, 19.5, 20.5, 24.5 } of abscissae γ_i, $i = 1, \ldots, 4$, at which the slope changes, and by the set { 1, 1.5, 2, 2.5 } of slopes μ_i, $i = 1, \ldots, 4$, in the various intervals.

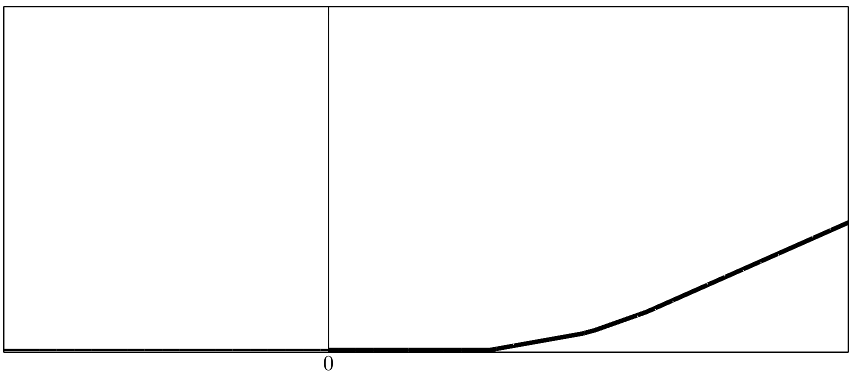


Figure 100: Optimal cost-to-go $J^{\circ}_{1,3,2}(t_4)$ in state $[1\ 3\ 2\ t_4]^T$.

Stage 4 – State $[1\ 3\ 1\ t_4]^T$ ($S18$)

In state $[1\ 3\ 1\ t_4]^T$ all jobs of class P_2 have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuos) decision variable τ only (which corresponds to the processing time $pt_{1,2}$), is

$$\alpha_{1,2}\ \max\{t_4 + st_{1,1} + \tau - dd_{1,2}\ ,\ 0\} + \beta_1\,(pt^{\mathrm{nom}}_1 - \tau) + sc_{1,1} + J^{\circ}_{2,3,1}(t_5)$$

that can be written as $f(pt_{1,2} + t_4) + g(pt_{1,2})$ being

$$f(pt_{1,2} + t_4) = 0.5 \cdot \max\{pt_{1,2} + t_4 - 24\ ,\ 0\} + J^{\circ}_{2,3,1}(pt_{1,2} + t_4)$$

$$g(pt_{1,2}) = \begin{cases} 8 - pt_{1,2} & pt_{1,2} \in [4, 8) \\ 0 & pt_{1,2} \notin [4, 8) \end{cases}$$

The function $pt^{\circ}_{1,2}(t_4) = \arg\min_{pt_{1,2}}\{f(pt_{1,2} + t_4) + g(pt_{1,2})\}$, with $4 \leq pt_{1,2} \leq 8$, is determined by applying lemma 1. It is

$$pt^{\circ}_{1,2}(t_4) = x_{\mathrm{e}}(t_4) \qquad \text{with} \quad x_{\mathrm{e}}(t_4) = \begin{cases} 8 & t_4 < 13 \\ -t_4 + 21 & 13 \leq t_4 < 17 \\ 4 & t_4 \geq 17 \end{cases}$$

Taking into account the mandatory decision about the class of the next job to be executed, the optimal control strategies for this state are

$$\delta^{\circ}_1(1, 3, 1, t_4) = 1 \quad \forall\, t_4 \qquad \delta^{\circ}_2(1, 3, 1, t_4) = 0 \quad \forall\, t_4$$

$$\tau^{\circ}(1, 3, 1, t_4) = \begin{cases} 8 & t_4 < 13 \\ -t_4 + 21 & 13 \leq t_4 < 17 \\ 4 & t_4 \geq 17 \end{cases}$$

The optimal control strategy $\tau^{\circ}(1, 3, 1, t_4)$ is illustrated in figure 101.

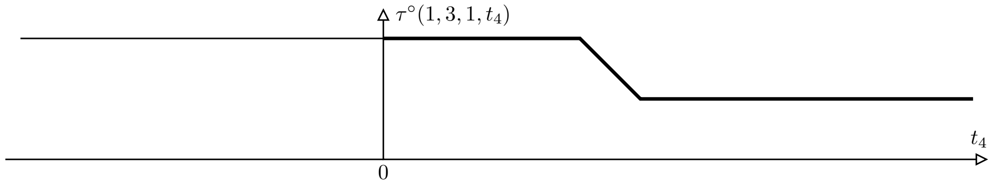


Figure 101: Optimal control strategy $\tau^{\circ}(1, 3, 1, t_4)$ in state $[1\ 3\ 1\ t_4]^T$.

The optimal cost-to-go $J^{\circ}_{1,3,1}(t_4) = f(pt^{\circ}_{1,2}(t_4) + t_4) + g(pt^{\circ}_{1,2}(t_4))$, illustrated in figure 102, is provided by lemma 2. It is specified by the initial value 0, by the set { 13, 20, 21, 25 } of abscissae γ_i, $i = 1, \ldots, 4$, at which the slope changes, and by the set { 1, 1.5, 2, 2.5 } of slopes μ_i, $i = 1, \ldots, 4$, in the various intervals.

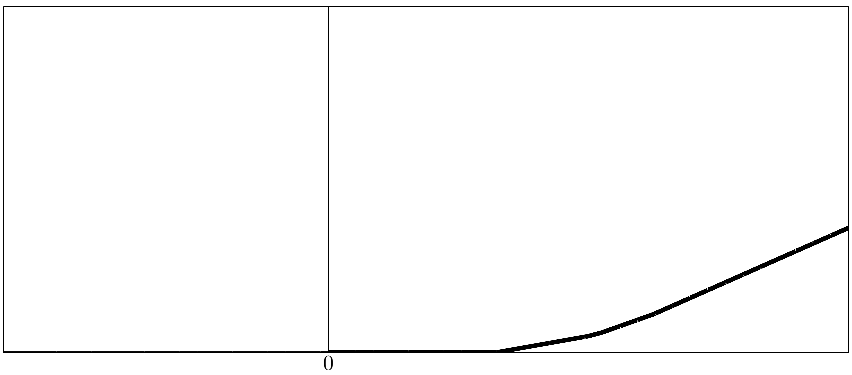


Figure 102: Optimal cost-to-go $J^{\circ}_{1,3,1}(t_4)$ in state $[1\ 3\ 1\ t_4]^T$.

Stage 4 – State $[2\ 2\ 2\ t_4]^T$ ($S17$)

In state $[2\ 2\ 2\ t_4]^T$, the cost function to be minimized, with respect to the (continuos) decision variable τ and to the (binary) decision variables δ_1 and δ_2 is

$$\delta_1 \big[\alpha_{1,3}\ \max\{t_4 + st_{2,1} + \tau - dd_{1,3}\ , 0\} + \beta_1\,(pt^{\mathrm{nom}}_1 - \tau) + sc_{2,1} + J^{\circ}_{3,2,1}(t_5)\big] +$$
$$+\ \delta_2 \big[\alpha_{2,3}\ \max\{t_4 + st_{2,2} + \tau - dd_{2,3}\ , 0\} + \beta_2\,(pt^{\mathrm{nom}}_2 - \tau) + sc_{2,2} + J^{\circ}_{2,3,2}(t_5)\big]$$

Case i) in which it is assumed $\delta_1 = 1$ (and $\delta_2 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{1,3}$, the following function

$$\alpha_{1,3}\ \max\{t_4 + st_{2,1} + \tau - dd_{1,3}\ , 0\} + \beta_1\,(pt^{\mathrm{nom}}_1 - \tau) + sc_{2,1} + J^{\circ}_{3,2,1}(t_5)$$

that can be written as $f(pt_{1,3}+t_4)+g(pt_{1,3})$ being

$$f(pt_{1,3}+t_4) = 1.5\cdot\max\{pt_{1,3}+t_4-28.5\,,\,0\}+1+J^\circ_{3,2,1}(pt_{1,3}+t_4+0.5)$$

$$g(pt_{1,3}) = \begin{cases} 8-pt_{1,3} & pt_{1,3}\in[4,8) \\ 0 & pt_{1,3}\notin[4,8) \end{cases}$$

The function $pt^\circ_{1,3}(t_4) = \arg\min_{pt_{1,3}}\{f(pt_{1,3}+t_4)+g(pt_{1,3})\}$, with $4\le pt_{1,3}\le 8$, is determined by applying lemma 1. It is (see figure 103)

$$pt^\circ_{1,3}(t_4) = \begin{cases} x_\mathrm{s}(t_4) & t_4<15.5 \\ x_\mathrm{e}(t_4) & t_4\ge 15.5 \end{cases} \qquad \text{with} \quad x_\mathrm{s}(t_4) = \begin{cases} 8 & t_4<14.5 \\ -t_4+22.5 & 14.5\le t_4<15.5 \end{cases},$$

$$\text{and} \quad x_\mathrm{e}(t_4) = \begin{cases} 8 & 15.5\le t_4<20.5 \\ -t_4+28.5 & 20.5\le t_4<24.5 \\ 4 & t_4\ge 24.5 \end{cases}$$

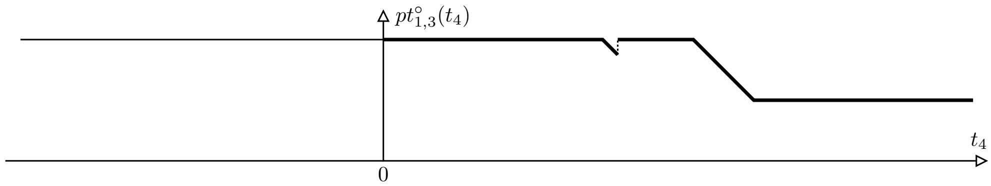


Figure 103: Optimal processing time $pt^\circ_{1,3}(t_4)$, under the assumption $\delta_1=1$ in state $[2\ 2\ 2\ t_4]^T$.

The conditioned cost-to-go $J^\circ_{2,2,2}(t_4\mid\delta_1=1) = f(pt^\circ_{1,3}(t_4)+t_4)+g(pt^\circ_{1,3}(t_4))$, illustrated in figure 105, is provided by lemma 2. It is specified by the initial value 1.5, by the set { 14.5, 15.5, 17, 20.5, 24.5, 26.5 } of abscissae γ_i, $i=1,\dots,6$, at which the slope changes, and by the set { 1, 0, 0.5, 1, 2, 3 } of slopes μ_i, $i=1,\dots,6$, in the various intervals.

Case ii) in which it is assumed $\delta_2=1$ (and $\delta_1=0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{2,3}$, the following function

$$\alpha_{2,3}\,\max\{t_4+st_{2,2}+\tau-dd_{2,3}\,,\,0\}+\beta_2\,(pt_2^{\mathrm{nom}}-\tau)+sc_{2,2}+J^\circ_{2,3,2}(t_5)$$

that can be written as $f(pt_{2,3}+t_4)+g(pt_{2,3})$ being

$$f(pt_{2,3}+t_4) = \max\{pt_{2,3}+t_4-38\,,\,0\}+J^\circ_{2,3,2}(pt_{2,3}+t_4)$$

$$g(pt_{2,3}) = \begin{cases} 1.5\cdot(6-pt_{2,3}) & pt_{2,3}\in[4,6) \\ 0 & pt_{2,3}\notin[4,6) \end{cases}$$

The function $pt^\circ_{2,3}(t_4) = \arg\min_{pt_{2,3}}\{f(pt_{2,3}+t_4)+g(pt_{2,3})\}$, with $4\le pt_{2,3}\le 6$, is determined by applying lemma 1. It is (see figure 104)

$$pt^\circ_{2,3}(t_4) = x_\mathrm{e}(t_4) \qquad \text{with} \quad x_\mathrm{e}(t_4) = \begin{cases} 6 & t_4<18.5 \\ -t_4+24.5 & 18.5\le t_4<20.5 \\ 4 & t_4\ge 20.5 \end{cases}$$

The conditioned cost-to-go $J^\circ_{2,2,2}(t_4\mid\delta_2=1) = f(pt^\circ_{2,3}(t_4)+t_4)+g(pt^\circ_{2,3}(t_4))$, illustrated in figure 105, is provided by lemma 2. It is specified by the initial value 1, by the set { 14.5, 18.5, 24.5, 34 } of abscissae γ_i, $i=1,\dots,4$, at which the slope changes, and by the set { 1, 1.5, 2, 3 } of slopes μ_i, $i=1,\dots,4$, in the various intervals.

In order to find the optimal cost-to-go $J^\circ_{2,2,2}(t_4)$, it is necessary to carry out the following minimization

$$J^\circ_{2,2,2}(t_4) = \min\left\{J^\circ_{2,2,2}(t_4\mid\delta_1=1)\,,\,J^\circ_{2,2,2}(t_4\mid\delta_2=1)\right\}$$

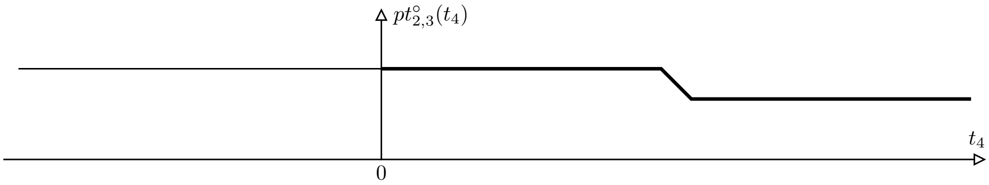


Figure 104: Optimal processing time $pt^{\circ}_{2,3}(t_4)$, under the assumption $\delta_2 = 1$ in state $[2\ 2\ 2\ t_4]^T$.

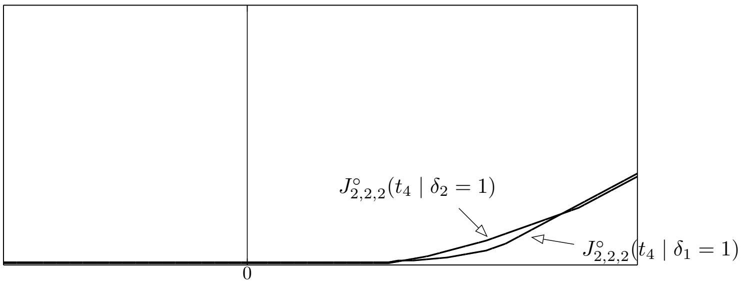


Figure 105: Conditioned costs-to-go $J^{\circ}_{2,2,2}(t_4 \mid \delta_1 = 1)$ and $J^{\circ}_{2,2,2}(t_4 \mid \delta_2 = 1)$ in state $[2\ 2\ 2\ t_4]^T$.

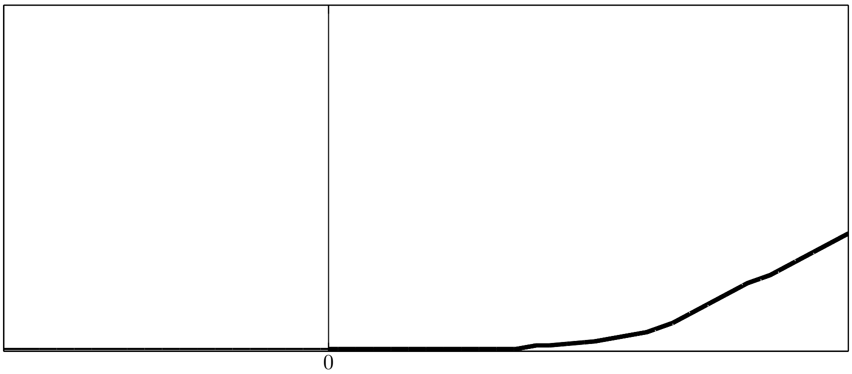


Figure 106: Optimal cost-to-go $J^{\circ}_{2,2,2}(t_4)$ in state $[2\ 2\ 2\ t_4]^T$.

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 106.

The function $J^{\circ}_{2,2,2}(t_4)$ is specified by the initial value 1, by the set { 14.5, 16, 17, 20.5, 24.5, 26.5, 32.25, 34 } of abscissae γ_i, $i = 1, \ldots, 8$, at which the slope changes, and by the set { 1, 0, 0.5, 1, 2, 3, 2, 3 } of slopes μ_i, $i = 1, \ldots, 8$, in the various intervals.

Since $J^{\circ}_{2,2,2}(t_4 \mid \delta_1 = 1)$ is the minimum in $[16, 32.25)$, and $J^{\circ}_{2,2,2}(t_4 \mid \delta_2 = 1)$ is the minimum in $(-\infty, 16)$ and in $[32.25, +\infty)$, the optimal control strategies for this state are

$$\delta^{\circ}_1(2,2,2,t_4) = \begin{cases} 0 & t_4 < 16 \\ 1 & 16 \le t_4 < 32.25 \\ 0 & t_4 \ge 32.25 \end{cases} \qquad \delta^{\circ}_2(2,2,2,t_4) = \begin{cases} 1 & t_4 < 16 \\ 0 & 16 \le t_4 < 32.25 \\ 1 & t_4 \ge 32.25 \end{cases}$$

$$\tau^{\circ}(2,2,2,t_4) = \begin{cases} 6 & t_4 < 16 \\ 8 & 16 \le t_4 < 20.5 \\ -t_4 + 28.5 & 20.5 \le t_4 < 24.5 \\ 4 & t_4 \ge 24.5 \end{cases}$$

The optimal control strategy $\tau^{\circ}(2,2,2,t_4)$ is illustrated in figure 107.

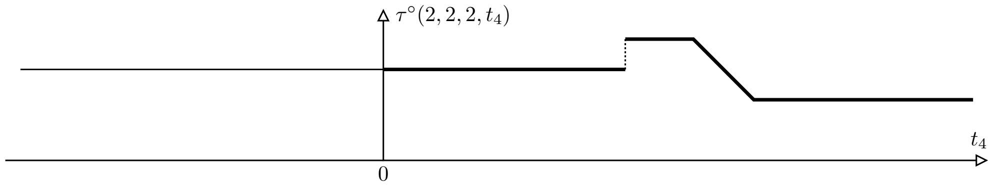


Figure 107: Optimal control strategy $\tau^\circ(2,2,2,t_4)$ in state $[2\ 2\ 2\ t_4]^T$.

Stage 4 – State $[\mathbf{2\ 2\ 1}\ \boldsymbol{t_4}]^{\boldsymbol{T}}$ ($S16$)

In state $[2\ 2\ 1\ t_4]^T$, the cost function to be minimized, with respect to the (continuos) decision variable τ and to the (binary) decision variables δ_1 and δ_2 is

$$\delta_1\big[\alpha_{1,3}\ \max\{t_4 + st_{1,1} + \tau - dd_{1,3}\ ,\ 0\} + \beta_1\,(pt_1^{\mathrm{nom}} - \tau) + sc_{1,1} + J^\circ_{3,2,1}(t_5)\big] + \\ + \delta_2\big[\alpha_{2,3}\ \max\{t_4 + st_{1,2} + \tau - dd_{2,3}\ ,\ 0\} + \beta_2\,(pt_2^{\mathrm{nom}} - \tau) + sc_{1,2} + J^\circ_{2,3,2}(t_5)\big]$$

Case i) in which it is assumed $\delta_1 = 1$ (and $\delta_2 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{1,3}$, the following function

$$\alpha_{1,3}\ \max\{t_4 + st_{1,1} + \tau - dd_{1,3}\ ,\ 0\} + \beta_1\,(pt_1^{\mathrm{nom}} - \tau) + sc_{1,1} + J^\circ_{3,2,1}(t_5)$$

that can be written as $f(pt_{1,3} + t_4) + g(pt_{1,3})$ being

$$f(pt_{1,3} + t_4) = 1.5 \cdot \max\{pt_{1,3} + t_4 - 29\ ,\ 0\} + J^\circ_{3,2,1}(pt_{1,3} + t_4)$$

$$g(pt_{1,3}) = \begin{cases} 8 - pt_{1,3} & pt_{1,3} \in [4,8) \\ 0 & pt_{1,3} \notin [4,8) \end{cases}$$

The function $pt^\circ_{1,3}(t_4) = \arg\min_{pt_{1,3}}\{f(pt_{1,3} + t_4) + g(pt_{1,3})\}$, with $4 \le pt_{1,3} \le 8$, is determined by applying lemma 1. It is (see figure 108)

$$pt^\circ_{1,3}(t_4) = \begin{cases} x_{\mathrm{s}}(t_4) & t_4 < 16 \\ x_{\mathrm{e}}(t_4) & t_4 \ge 16 \end{cases} \qquad \text{with} \quad x_{\mathrm{s}}(t_4) = \begin{cases} 8 & t_4 < 15 \\ -t_4 + 23 & 15 \le t_4 < 16 \end{cases} ,$$

$$\text{and} \quad x_{\mathrm{e}}(t_4) = \begin{cases} 8 & 16 \le t_4 < 21 \\ -t_4 + 29 & 21 \le t_4 < 25 \\ 4 & t_4 \ge 25 \end{cases}$$

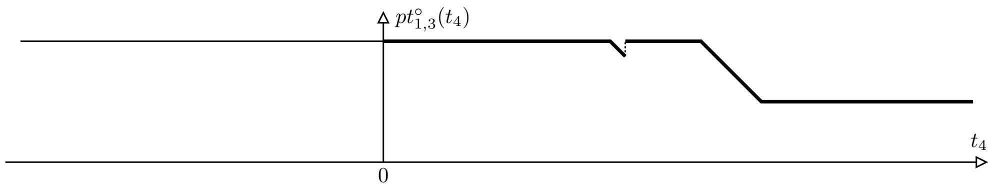


Figure 108: Optimal processing time $pt^\circ_{1,3}(t_4)$, under the assumption $\delta_1 = 1$ in state $[2\ 2\ 1\ t_4]^T$.

The conditioned cost-to-go $J^\circ_{2,2,1}(t_4 \mid \delta_1 = 1) = f(pt^\circ_{1,3}(t_4) + t_4) + g(pt^\circ_{1,3}(t_4))$, illustrated in figure 110, is provided by lemma 2. It is specified by the initial value 1.5, by the set { 15, 16, 17.5, 21, 25, 27 } of abscissae γ_i, $i = 1, \ldots, 6$, at which the slope changes, and by the set { 1, 0, 0.5, 1, 2, 3 } of slopes μ_i, $i = 1, \ldots, 6$, in the various intervals.

Case ii) in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{2,3}$, the following function

$$\alpha_{2,3}\ \max\{t_4 + st_{1,2} + \tau - dd_{2,3}\ ,\ 0\} + \beta_2\,(pt_2^{\mathrm{nom}} - \tau) + sc_{1,2} + J^\circ_{2,3,2}(t_5)$$

that can be written as $f(pt_{2,3}+t_4)+g(pt_{2,3})$ being

$$f(pt_{2,3}+t_4) = \max\{pt_{2,3}+t_4-37\,,\,0\} + 0.5 + J^{\circ}_{2,3,2}(pt_{2,3}+t_4+1)$$

$$g(pt_{2,3}) = \begin{cases} 1.5\cdot(6-pt_{2,3}) & pt_{2,3}\in[4,6) \\ 0 & pt_{2,3}\notin[4,6) \end{cases}$$

The function $pt^{\circ}_{2,3}(t_4) = \arg\min_{pt_{2,3}}\{f(pt_{2,3}+t_4)+g(pt_{2,3})\}$, with $4 \le pt_{2,3} \le 6$, is determined by applying lemma 1. It is (see figure 109)

$$pt^{\circ}_{2,3}(t_4) = x_{\mathrm{e}}(t_4) \qquad \text{with} \quad x_{\mathrm{e}}(t_4) = \begin{cases} 6 & t_4 < 17.5 \\ -t_4+23.5 & 17.5 \le t_4 < 19.5 \\ 4 & t_4 \ge 19.5 \end{cases}$$

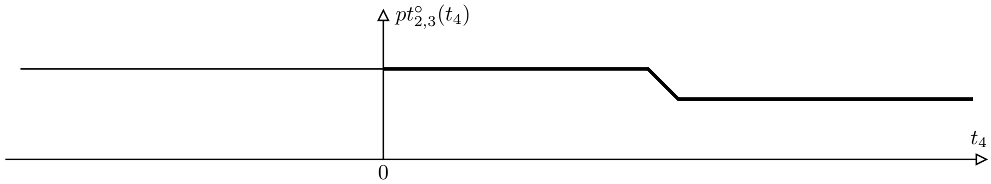


Figure 109: Optimal processing time $pt^{\circ}_{2,3}(t_4)$, under the assumption $\delta_2 = 1$ in state $[2\ 2\ 1\ t_4]^T$.

The conditioned cost-to-go $J^{\circ}_{2,2,1}(t_4 \mid \delta_2 = 1) = f(pt^{\circ}_{2,3}(t_4)+t_4) + g(pt^{\circ}_{2,3}(t_4))$, illustrated in figure 110, is provided by lemma 2. It is specified by the initial value 1.5, by the set { 13.5, 17.5, 23.5, 33 } of abscissae γ_i, $i = 1,\ldots,4$, at which the slope changes, and by the set { 1, 1.5, 2, 3 } of slopes μ_i, $i = 1,\ldots,4$, in the various intervals.

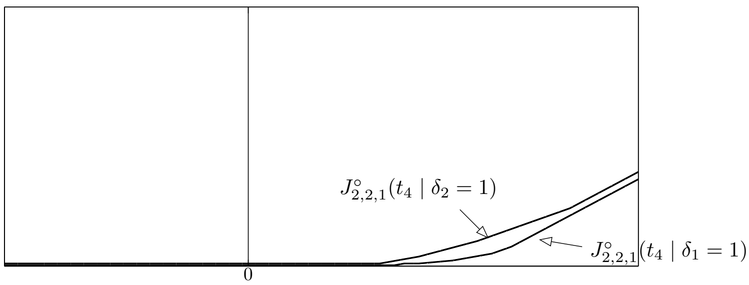


Figure 110: Conditioned costs-to-go $J^{\circ}_{2,2,1}(t_4 \mid \delta_1 = 1)$ and $J^{\circ}_{2,2,1}(t_4 \mid \delta_2 = 1)$ in state $[2\ 2\ 1\ t_4]^T$.

In order to find the optimal cost-to-go $J^{\circ}_{2,2,1}(t_4)$, it is necessary to carry out the following minimization

$$J^{\circ}_{2,2,1}(t_4) = \min\left\{J^{\circ}_{2,2,1}(t_4 \mid \delta_1 = 1)\,,\, J^{\circ}_{2,2,1}(t_4 \mid \delta_2 = 1)\right\}$$

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 111.

The function $J^{\circ}_{2,2,1}(t_4)$ is specified by the initial value 0.5, by the set { 15, 16, 17.5, 21, 25, 27 } of abscissae γ_i, $i = 1,\ldots,6$, at which the slope changes, and by the set { 1, 0, 0.5, 1, 2, 3 } of slopes μ_i, $i = 1,\ldots,6$, in the various intervals.

Since $J^{\circ}_{2,2,1}(t_4 \mid \delta_1 = 1)$ is always the minimum (see again figure 110), the optimal control strategies for this state are

$$\delta^{\circ}_1(2,2,1,t_4) = 1 \quad \forall\, t_4 \qquad \delta^{\circ}_2(2,2,1,t_4) = 0 \quad \forall\, t_4$$

$$\tau^{\circ}(2,2,1,t_4) = \begin{cases} 8 & t_4 < 15 \\ -t_4+23 & 15 \le t_4 < 16 \\ 8 & 16 \le t_4 < 21 \\ -t_4+29 & 21 \le t_4 < 25 \\ 4 & t_4 \ge 25 \end{cases}$$

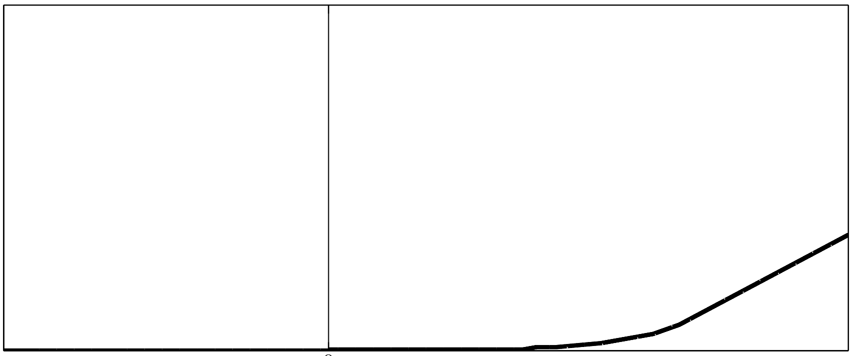


Figure 111: Optimal cost-to-go $J^\circ_{2,2,1}(t_4)$ in state $[2\ 2\ 1\ t_4]^T$.

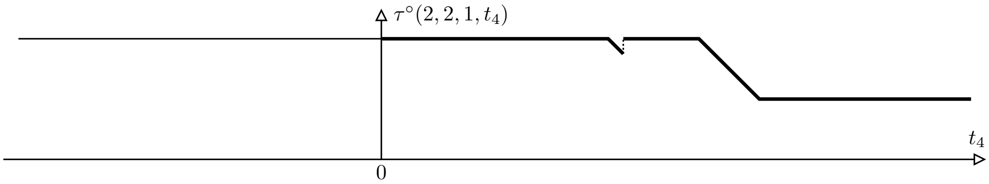


Figure 112: Optimal control strategy $\tau^\circ(2,2,1,t_4)$ in state $[2\ 2\ 1\ t_4]^T$.

The optimal control strategy $\tau^\circ(2,2,1,t_4)$ is illustrated in figure 112.

Stage 4 – State $[3\ 1\ 2\ t_4]^T$ ($S15$)

In state $[3\ 1\ 2\ t_4]^T$, the cost function to be minimized, with respect to the (continuos) decision variable τ and to the (binary) decision variables δ_1 and δ_2 is

$$\delta_1\big[\alpha_{1,4}\ \max\{t_4 + st_{2,1} + \tau - dd_{1,4}\ ,\ 0\} + \beta_1\,(pt_1^{\text{nom}} - \tau) + sc_{2,1} + J^\circ_{4,1,1}(t_5)\big] +$$
$$+\ \delta_2\big[\alpha_{2,2}\ \max\{t_4 + st_{2,2} + \tau - dd_{2,2}\ ,\ 0\} + \beta_2\,(pt_2^{\text{nom}} - \tau) + sc_{2,2} + J^\circ_{3,2,2}(t_5)\big]$$

Case i) in which it is assumed $\delta_1 = 1$ (and $\delta_2 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{1,4}$, the following function

$$\alpha_{1,4}\ \max\{t_4 + st_{2,1} + \tau - dd_{1,4}\ ,\ 0\} + \beta_1\,(pt_1^{\text{nom}} - \tau) + sc_{2,1} + J^\circ_{4,1,1}(t_5)$$

that can be written as $f(pt_{1,4} + t_4) + g(pt_{1,4})$ being

$$f(pt_{1,4} + t_4) = 0.5 \cdot \max\{pt_{1,4} + t_4 - 40.5\ ,\ 0\} + 1 + J^\circ_{4,1,1}(pt_{1,4} + t_4 + 0.5)$$

$$g(pt_{1,4}) = \begin{cases} 8 - pt_{1,4} & pt_{1,4} \in [4,8) \\ 0 & pt_{1,4} \notin [4,8) \end{cases}$$

The function $pt^\circ_{1,4}(t_4) = \arg\min_{pt_{1,4}}\{f(pt_{1,4} + t_4) + g(pt_{1,4})\}$, with $4 \le pt_{1,4} \le 8$, is determined by applying lemma 1. It is (see figure 113)

$$pt^\circ_{1,4}(t_4) = x_{\text{e}}(t_4) \qquad \text{with} \quad x_{\text{e}}(t_4) = \begin{cases} 8 & t_4 < 8.5 \\ -t_4 + 16.5 & 8.5 \le t_4 < 12.5 \\ 4 & t_4 \ge 12.5 \end{cases}$$

The conditioned cost-to-go $J^\circ_{3,1,2}(t_4 \mid \delta_1 = 1) = f(pt^\circ_{1,4}(t_4) + t_4) + g(pt^\circ_{1,4}(t_4))$, illustrated in figure 115, is provided by lemma 2. It is specified by the initial value 1.5, by the set { 8.5, 20.5, 22.5, 36.5 } of abscissae γ_i, $i = 1, \ldots, 4$, at which the slope changes, and by the set { 1, 1.5, 2, 2.5 } of slopes μ_i, $i = 1, \ldots, 4$, in the various intervals.

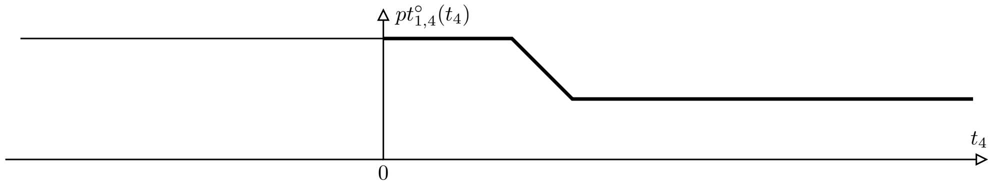


Figure 113: Optimal processing time $pt^{\circ}_{1,4}(t_4)$, under the assumption $\delta_1 = 1$ in state $[3\ 1\ 2\ t_4]^T$.

Case ii) in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{2,2}$, the following function

$$\alpha_{2,2}\ \max\{t_4 + st_{2,2} + \tau - dd_{2,2}\ ,\ 0\} + \beta_2\,(pt_2^{\mathrm{nom}} - \tau) + sc_{2,2} + J^{\circ}_{3,2,2}(t_5)$$

that can be written as $f(pt_{2,2} + t_4) + g(pt_{2,2})$ being

$$f(pt_{2,2} + t_4) = \max\{pt_{2,2} + t_4 - 24\ ,\ 0\} + J^{\circ}_{3,2,2}(pt_{2,2} + t_4)$$

$$g(pt_{2,2}) = \begin{cases} 1.5 \cdot (6 - pt_{2,2}) & pt_{2,2} \in [4, 6) \\ 0 & pt_{2,2} \notin [4, 6) \end{cases}$$

The function $pt^{\circ}_{2,2}(t_4) = \arg\min_{pt_{2,2}}\{f(pt_{2,2} + t_4) + g(pt_{2,2})\}$, with $4 \le pt_{2,2} \le 6$, is determined by applying lemma 1. It is (see figure 114)

$$pt^{\circ}_{2,2}(t_4) = x_{\mathrm{e}}(t_4) \qquad \text{with} \quad x_{\mathrm{e}}(t_4) = \begin{cases} 6 & t_4 < 20.5 \\ -t_4 + 26.5 & 20.5 \le t_4 < 22.5 \\ 4 & t_4 \ge 22.5 \end{cases}$$

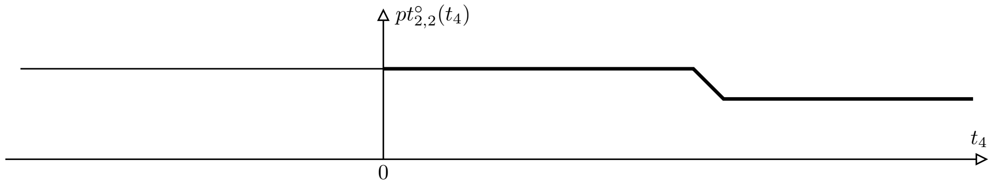


Figure 114: Optimal processing time $pt^{\circ}_{2,2}(t_4)$, under the assumption $\delta_2 = 1$ in state $[3\ 1\ 2\ t_4]^T$.

The conditioned cost-to-go $J^{\circ}_{3,1,2}(t_4 \mid \delta_2 = 1) = f(pt^{\circ}_{2,2}(t_4) + t_4) + g(pt^{\circ}_{2,2}(t_4))$, illustrated in figure 115, is provided by lemma 2. It is specified by the initial value 1, by the set { 18, 20.5, 28 } of abscissae γ_i, $i = 1, \ldots, 3$, at which the slope changes, and by the set { 1, 1.5, 2.5 } of slopes μ_i, $i = 1, \ldots, 3$, in the various intervals.

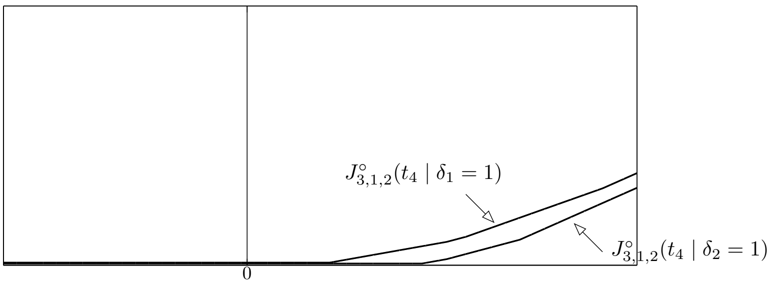


Figure 115: Conditioned costs-to-go $J^{\circ}_{3,1,2}(t_4 \mid \delta_1 = 1)$ and $J^{\circ}_{3,1,2}(t_4 \mid \delta_2 = 1)$ in state $[3\ 1\ 2\ t_4]^T$.

In order to find the optimal cost-to-go $J^{\circ}_{3,1,2}(t_4)$, it is necessary to carry out the following minimization

$$J^{\circ}_{3,1,2}(t_4) = \min\left\{J^{\circ}_{3,1,2}(t_4 \mid \delta_1 = 1)\ ,\ J^{\circ}_{3,1,2}(t_4 \mid \delta_2 = 1)\right\}$$

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 116.

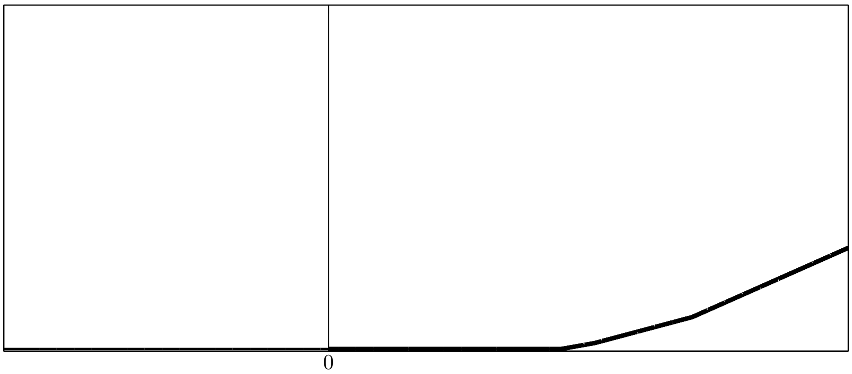


Figure 116: Optimal cost-to-go $J^{\circ}_{3,1,2}(t_4)$ in state $[3\ 1\ 2\ t_4]^T$.

The function $J^{\circ}_{3,1,2}(t_4)$ is specified by the initial value 1, by the set { 18, 20.5, 28 } of abscissae γ_i, $i = 1, \ldots, 3$, at which the slope changes, and by the set { 1, 1.5, 2.5 } of slopes μ_i, $i = 1, \ldots, 3$, in the various intervals.

Since $J^{\circ}_{3,1,2}(t_4 \mid \delta_2 = 1)$ is always the minimum (see again figure 115), the optimal control strategies for this state are

$$\delta_1^{\circ}(3,1,2,t_4) = 0 \quad \forall\, t_4 \qquad \delta_2^{\circ}(3,1,2,t_4) = 1 \quad \forall\, t_4$$

$$\tau^{\circ}(3,1,2,t_4) = \begin{cases} 6 & t_4 < 20.5 \\ -t_4 + 26.5 & 20.5 \le t_4 < 22.5 \\ 4 & t_4 \ge 22.5 \end{cases}$$

The optimal control strategy $\tau^{\circ}(3,1,2,t_4)$ is illustrated in figure 117.

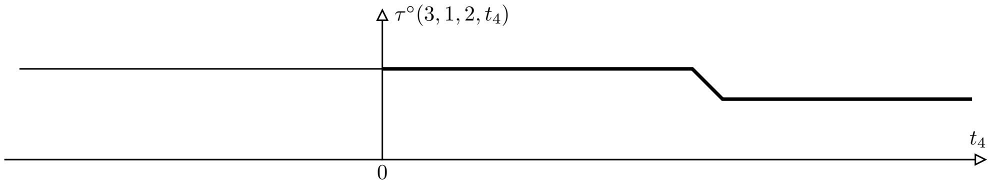


Figure 117: Optimal control strategy $\tau^{\circ}(3,1,2,t_4)$ in state $[3\ 1\ 2\ t_4]^T$.

Stage 4 – State $[3\ 1\ 1\ t_4]^T$ ($S14$)

In state $[3\ 1\ 1\ t_4]^T$, the cost function to be minimized, with respect to the (continuos) decision variable τ and to the (binary) decision variables δ_1 and δ_2 is

$$\delta_1 \big[\alpha_{1,4}\, \max\{t_4 + st_{1,1} + \tau - dd_{1,4}\,,\, 0\} + \beta_1\, (pt_1^{\text{nom}} - \tau) + sc_{1,1} + J^{\circ}_{4,1,1}(t_5)\big] +$$
$$+\, \delta_2 \big[\alpha_{2,2}\, \max\{t_4 + st_{1,2} + \tau - dd_{2,2}\,,\, 0\} + \beta_2\, (pt_2^{\text{nom}} - \tau) + sc_{1,2} + J^{\circ}_{3,2,2}(t_5)\big]$$

Case i) in which it is assumed $\delta_1 = 1$ (and $\delta_2 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{1,4}$, the following function

$$\alpha_{1,4}\, \max\{t_4 + st_{1,1} + \tau - dd_{1,4}\,,\, 0\} + \beta_1\, (pt_1^{\text{nom}} - \tau) + sc_{1,1} + J^{\circ}_{4,1,1}(t_5)$$

that can be written as $f(pt_{1,4} + t_4) + g(pt_{1,4})$ being

$$f(pt_{1,4} + t_4) = 0.5 \cdot \max\{pt_{1,4} + t_4 - 41\,,\, 0\} + J^{\circ}_{4,1,1}(pt_{1,4} + t_4)$$

$$g(pt_{1,4}) = \begin{cases} 8 - pt_{1,4} & pt_{1,4} \in [4,8) \\ 0 & pt_{1,4} \notin [4,8) \end{cases}$$

The function $pt^{\circ}_{1,4}(t_4) = \arg\min_{pt_{1,4}} \{f(pt_{1,4} + t_4) + g(pt_{1,4})\}$, with $4 \le pt_{1,4} \le 8$, is determined by applying lemma 1. It is (see figure 118)

$$pt^{\circ}_{1,4}(t_4) = x_{\mathrm{e}}(t_4) \qquad \text{with} \quad x_{\mathrm{e}}(t_4) = \begin{cases} 8 & t_4 < 9 \\ -t_4 + 17 & 9 \le t_4 < 13 \\ 4 & t_4 \ge 13 \end{cases}$$

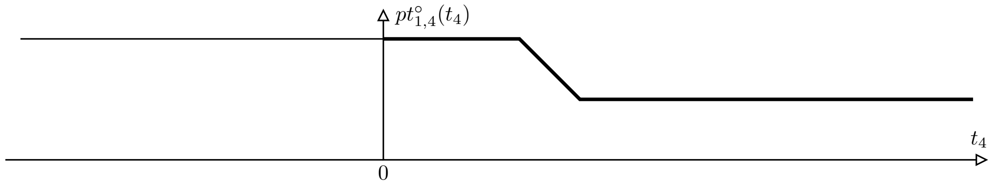


Figure 118: Optimal processing time $pt^{\circ}_{1,4}(t_4)$, under the assumption $\delta_1 = 1$ in state $[3\ 1\ 1\ t_4]^T$.

The conditioned cost-to-go $J^{\circ}_{3,1,1}(t_4 \mid \delta_1 = 1) = f(pt^{\circ}_{1,4}(t_4) + t_4) + g(pt^{\circ}_{1,4}(t_4))$, illustrated in figure 120, is provided by lemma 2. It is specified by the initial value 0.5, by the set { 9, 21, 23, 37 } of abscissae γ_i, $i = 1, \ldots, 4$, at which the slope changes, and by the set { 1, 1.5, 2, 2.5 } of slopes μ_i, $i = 1, \ldots, 4$, in the various intervals.

Case ii) in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{2,2}$, the following function

$$\alpha_{2,2} \max\{t_4 + st_{1,2} + \tau - dd_{2,2}\, ,\, 0\} + \beta_2\, (pt^{\mathrm{nom}}_2 - \tau) + sc_{1,2} + J^{\circ}_{3,2,2}(t_5)$$

that can be written as $f(pt_{2,2} + t_4) + g(pt_{2,2})$ being

$$f(pt_{2,2} + t_4) = \max\{pt_{2,2} + t_4 - 23\, ,\, 0\} + 0.5 + J^{\circ}_{3,2,2}(pt_{2,2} + t_4 + 1)$$

$$g(pt_{2,2}) = \begin{cases} 1.5 \cdot (6 - pt_{2,2}) & pt_{2,2} \in [4, 6) \\ 0 & pt_{2,2} \notin [4, 6) \end{cases}$$

The function $pt^{\circ}_{2,2}(t_4) = \arg\min_{pt_{2,2}} \{f(pt_{2,2} + t_4) + g(pt_{2,3})\}$, with $4 \le pt_{2,2} \le 6$, is determined by applying lemma 1. It is (see figure 119)

$$pt^{\circ}_{2,2}(t_4) = x_{\mathrm{e}}(t_4) \qquad \text{with} \quad x_{\mathrm{e}}(t_4) = \begin{cases} 6 & t_4 < 19.5 \\ -t_4 + 25.5 & 19.5 \le t_4 < 21.5 \\ 4 & t_4 \ge 21.5 \end{cases}$$

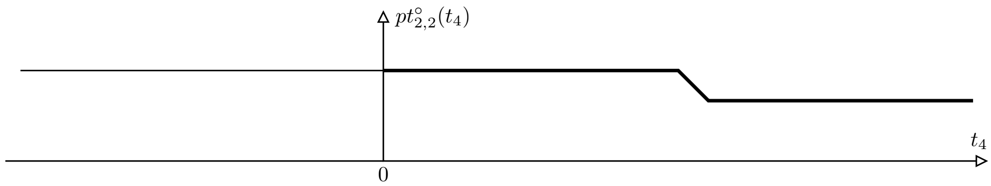


Figure 119: Optimal processing time $pt^{\circ}_{2,2}(t_4)$, under the assumption $\delta_2 = 1$ in state $[3\ 1\ 1\ t_4]^T$.

The conditioned cost-to-go $J^{\circ}_{3,1,1}(t_4 \mid \delta_2 = 1) = f(pt^{\circ}_{2,2}(t_4) + t_4) + g(pt^{\circ}_{2,2}(t_4))$, illustrated in figure 120, is provided by lemma 2. It is specified by the initial value 1, by the set { 17, 19.5, 27 } of abscissae γ_i, $i = 1, \ldots, 3$, at which the slope changes, and by the set { 1, 1.5, 2.5 } of slopes μ_i, $i = 1, \ldots, 3$, in the various intervals.

In order to find the optimal cost-to-go $J^{\circ}_{3,1,1}(t_4)$, it is necessary to carry out the following minimization

$$J^{\circ}_{3,1,1}(t_4) = \min\left\{J^{\circ}_{3,1,1}(t_4 \mid \delta_1 = 1)\, ,\, J^{\circ}_{3,1,1}(t_4 \mid \delta_2 = 1)\right\}$$

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 121.

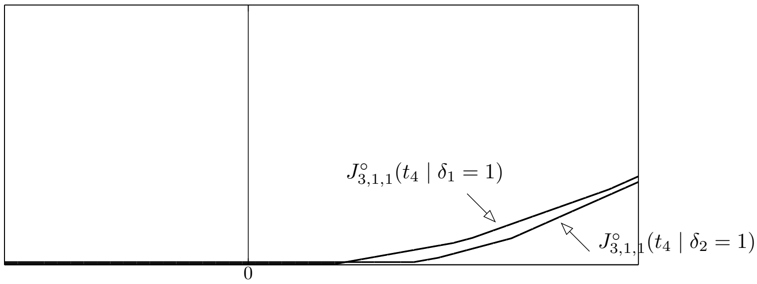


Figure 120: Conditioned costs-to-go $J^\circ_{3,1,1}(t_4 \mid \delta_1 = 1)$ and $J^\circ_{3,1,1}(t_4 \mid \delta_2 = 1)$ in state $[3\ 1\ 1\ t_4]^T$.

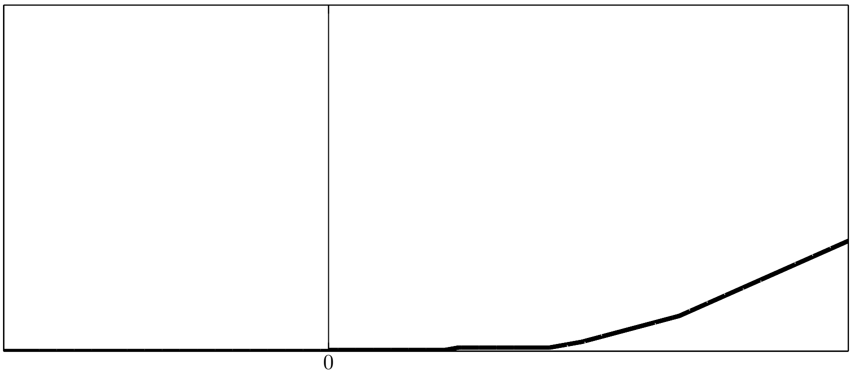


Figure 121: Optimal cost-to-go $J^\circ_{3,1,1}(t_4)$ in state $[3\ 1\ 1\ t_4]^T$.

The function $J^\circ_{3,1,1}(t_4)$ is specified by the initial value 0.5, by the set { 9, 10, 17, 19.5, 27 } of abscissae γ_i, $i = 1, \ldots, 5$, at which the slope changes, and by the set { 1, 0, 1, 1.5, 2.5 } of slopes μ_i, $i = 1, \ldots, 5$, in the various intervals.

Since $J^\circ_{3,1,1}(t_4 \mid \delta_1 = 1)$ is the minimum in $(-\infty, 10)$, and $J^\circ_{3,1,1}(t_4 \mid \delta_2 = 1)$ is the minimum in $[10, +\infty)$, the optimal control strategies for this state are

$$\delta_1^\circ(3,1,1,t_4) = \begin{cases} 1 & t_4 < 10 \\ 0 & t_4 \geq 10 \end{cases} \qquad \delta_2^\circ(3,1,1,t_4) = \begin{cases} 0 & t_4 < 10 \\ 1 & t_4 \geq 10 \end{cases}$$

$$\tau^\circ(3,1,1,t_4) = \begin{cases} 8 & t_4 < 9 \\ -t_4 + 17 & 9 \leq t_4 < 24.5 \\ 6 & 10 \leq t_4 < 19.5 \\ -t_4 + 25.5 & 19.5 \leq t_4 < 21.5 \\ 4 & t_5 \geq 21.5 \end{cases}$$

The optimal control strategy $\tau^\circ(3,1,1,t_4)$ is illustrated in figure 122.

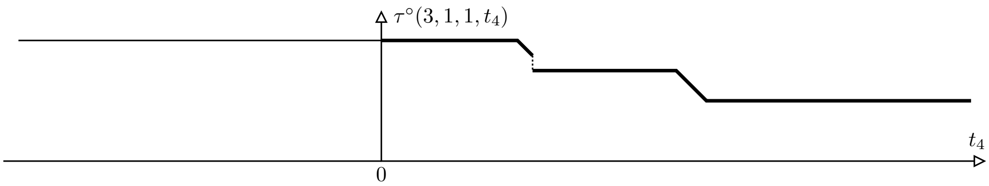


Figure 122: Optimal control strategy $\tau^\circ(3,1,1,t_4)$ in state $[3\ 1\ 1\ t_4]^T$.

Stage 4 – State $[4\ 0\ 1\ t_4]^T$ ($S13$)

In state $[4\ 0\ 1\ t_4]^T$ all jobs of class P_1 have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuos) decision variable τ only (which corresponds to the processing time $pt_{2,1}$), is

$$\alpha_{2,1}\ \max\{t_4 + st_{1,2} + \tau - dd_{2,1}\ ,\ 0\} + \beta_2\,(pt_2^{\text{nom}} - \tau) + sc_{1,2} + J^\circ_{4,1,2}(t_5)$$

that can be written as $f(pt_{2,1}+t_4)+g(pt_{2,1})$ being

$$f(pt_{2,1}+t_4) = 2\cdot\max\{pt_{2,1}+t_4-20\,,\,0\}+0.5+J^\circ_{4,1,2}(pt_{2,1}+t_4+1)$$

$$g(pt_{2,1}) = \begin{cases} 1.5\cdot(6-pt_{2,1}) & pt_{2,1}\in[4,6) \\ 0 & pt_{2,1}\notin[4,6) \end{cases}$$

The function $pt^\circ_{2,1}(t_4) = \arg\min_{pt_{2,1}}\{f(pt_{2,1}+t_4)+g(pt_{2,1})\}$, with $4\le pt_{2,1}\le 6$, is determined by applying lemma 1. It is

$$pt^\circ_{2,1}(t_4) = x_{\mathrm{e}}(t_4) \qquad \text{with} \quad x_{\mathrm{e}}(t_4) = \begin{cases} 6 & t_4<14 \\ -t_4+20 & 14\le t_4<16 \\ 4 & t_4\ge 16 \end{cases}$$

Taking into account the mandatory decision about the class of the next job to be executed, the optimal control strategies for this state are

$$\delta^\circ_1(4,0,1,t_4)=0 \quad \forall\, t_4 \qquad \delta^\circ_2(4,0,1,t_4)=1 \quad \forall\, t_4$$

$$\tau^\circ(4,0,1,t_4) = \begin{cases} 6 & t_4<14 \\ -t_4+20 & 14\le t_4<16 \\ 4 & t_4\ge 16 \end{cases}$$

The optimal control strategy $\tau^\circ(4,0,1,t_4)$ is illustrated in figure 123.

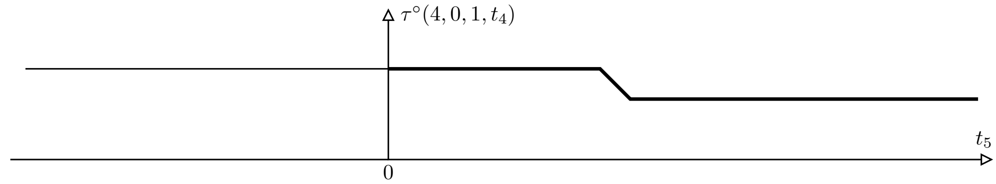


Figure 123: Optimal control strategy $\tau^\circ(4,0,1,t_4)$ in state $[4\ 0\ 1\ t_4]^T$.

The optimal cost-to-go $J^\circ_{4,0,1}(t_4) = f(pt^\circ_{2,1}(t_4)+t_4)+g(pt^\circ_{2,1}(t_4))$, illustrated in figure 124, is provided by lemma 2. It is specified by the initial value 0.5, by the set { 11, 14, 16, 21, 23 } of abscissae γ_i, $i=1,\dots,5$, at which the slope changes, and by the set { 1, 1.5, 3, 3.5, 4 } of slopes μ_i, $i=1,\dots,5$, in the various intervals.

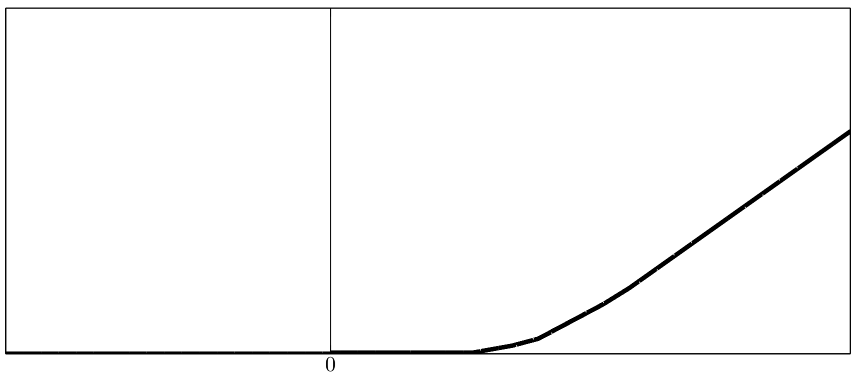


Figure 124: Optimal cost-to-go $J^\circ_{4,0,1}(t_4)$ in state $[4\ 0\ 1\ t_4]^T$.

Stage 3 – State $[\mathbf{0\ 3\ 2}\ \boldsymbol{t_3}]^{\boldsymbol{T}}$ ($S12$)

In state $[0\ 3\ 2\ t_3]^T$ all jobs of class P_2 have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuos) decision variable τ only (which corresponds to the processing time $pt_{1,1}$), is

$$\alpha_{1,1}\ \max\{t_3+st_{2,1}+\tau-dd_{1,1}\,,\,0\}+\beta_1\,(pt^{\mathrm{nom}}_1-\tau)+sc_{2,1}+J^\circ_{1,3,1}(t_4)$$

that can be written as $f(pt_{1,1}+t_3)+g(pt_{1,1})$ being

$$f(pt_{1,1}+t_3) = 0.75\cdot\max\{pt_{1,1}+t_3-18.5\,,\,0\}+1+J^\circ_{1,3,1}(pt_{1,1}+t_3+0.5)$$

$$g(pt_{1,1}) = \begin{cases} 8-pt_{1,1} & pt_{1,1}\in[4,8) \\ 0 & pt_{1,1}\notin[4,8) \end{cases}$$

The function $pt^\circ_{1,1}(t_3)=\arg\min_{pt_{1,1}}\{f(pt_{1,1}+t_3)+g(pt_{1,1})\}$, with $4\le pt_{1,1}\le 8$, is determined by applying lemma 1. It is

$$pt^\circ_{1,1}(t_3)=x_{\mathrm{e}}(t_3) \qquad \text{with} \quad x_{\mathrm{e}}(t_3)=\begin{cases} 8 & t_3<4.5 \\ -t_3+12.5 & 4.5\le t_3<8.5 \\ 4 & t_3\ge 8.5 \end{cases}$$

Taking into account the mandatory decision about the class of the next job to be executed, the optimal control strategies for this state are

$$\delta^\circ_1(0,3,2,t_3)=1 \quad \forall\, t_3 \qquad \delta^\circ_2(0,3,2,t_3)=0 \quad \forall\, t_3$$

$$\tau^\circ(0,3,2,t_3)=\begin{cases} 8 & t_3<4.5 \\ -t_3+12.5 & 4.5\le t_3<8.5 \\ 4 & t_3\ge 8.5 \end{cases}$$

The optimal control strategy $\tau^\circ(0,3,2,t_3)$ is illustrated in figure 125.

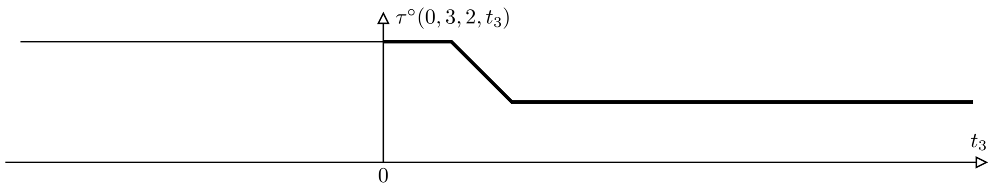


Figure 125: Optimal control strategy $\tau^\circ(0,3,2,t_3)$ in state $[0\ 3\ 2\ t_3]^T$.

The optimal cost-to-go $J^\circ_{0,3,2}(t_3) = f(pt^\circ_{1,1}(t_3)+t_3)+g(pt^\circ_{1,1}(t_3))$, illustrated in figure 126, is provided by lemma 2. It is specified by the initial value 1, by the set { 4.5, 14.5, 15.5, 16.5, 20.5 } of abscissae γ_i, $i=1,\ldots,5$, at which the slope changes, and by the set { 1, 1.75, 2.25, 2.75, 3.25 } of slopes μ_i, $i=1,\ldots,5$, in the various intervals.

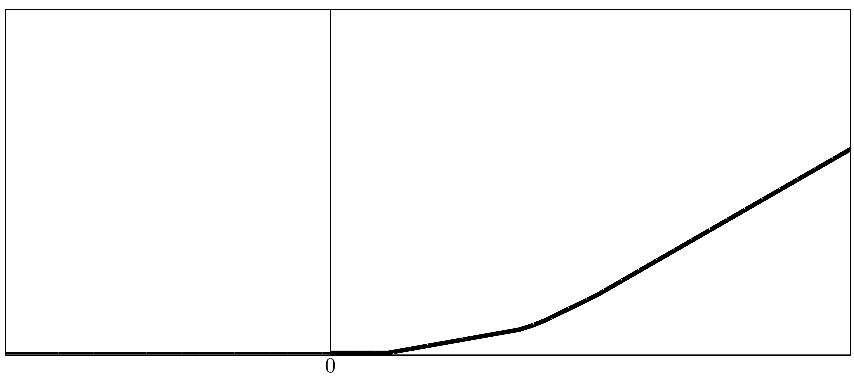


Figure 126: Optimal cost-to-go $J^\circ_{0,3,2}(t_3)$ in state $[0\ 3\ 2\ t_3]^T$.

Stage 3 – State $[\mathbf{1\ 2\ 2}\ \boldsymbol{t_3}]^T$ ($S11$)

In state $[1\ 2\ 2\ t_3]^T$, the cost function to be minimized, with respect to the (continuos) decision variable τ and to the (binary) decision variables δ_1 and δ_2 is

$$\delta_1\big[\alpha_{1,2}\ \max\{t_3+st_{2,1}+\tau-dd_{1,2}\,,\,0\}+\beta_1\,(pt_1^{\mathrm{nom}}-\tau)+sc_{2,1}+J^\circ_{2,2,1}(t_4)\big]+$$
$$+\,\delta_2\big[\alpha_{2,3}\ \max\{t_3+st_{2,2}+\tau-dd_{2,3}\,,\,0\}+\beta_2\,(pt_2^{\mathrm{nom}}-\tau)+sc_{2,2}+J^\circ_{1,3,2}(t_4)\big]$$

Case i) in which it is assumed $\delta_1 = 1$ (and $\delta_2 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{1,2}$, the following function

$$\alpha_{1,2}\ \max\{t_3 + st_{2,1} + \tau - dd_{1,2}\ ,\ 0\} + \beta_1\,(pt_1^{\mathrm{nom}} - \tau) + sc_{2,1} + J^\circ_{2,2,1}(t_4)$$

that can be written as $f(pt_{1,2} + t_3) + g(pt_{1,2})$ being

$$f(pt_{1,2} + t_3) = 0.5 \cdot \max\{pt_{1,2} + t_3 - 23.5\ ,\ 0\} + 1 + J^\circ_{2,2,1}(pt_{1,2} + t_3 + 0.5)$$

$$g(pt_{1,2}) = \begin{cases} 8 - pt_{1,2} & pt_{1,2} \in [4, 8) \\ 0 & pt_{1,2} \notin [4, 8) \end{cases}$$

The function $pt^\circ_{1,2}(t_3) = \arg\min_{pt_{1,2}}\{f(pt_{1,2} + t_3) + g(pt_{1,2})\}$, with $4 \le pt_{1,2} \le 8$, is determined by applying lemma 1. It is (see figure 127)

$$pt^\circ_{1,2}(t_3) = \begin{cases} x_{\mathrm{s}}(t_3) & t_3 < 7.5 \\ x_{\mathrm{e}}(t_3) & t_3 \ge 7.5 \end{cases} \qquad \text{with} \quad x_{\mathrm{s}}(t_3) = \begin{cases} 8 & t_3 < 6.5 \\ -t_3 + 14.5 & 6.5 \le t_3 < 7.5 \end{cases},$$

$$\text{and} \quad x_{\mathrm{e}}(t_3) = \begin{cases} 8 & 7.5 \le t_3 < 12.5 \\ -t_3 + 20.5 & 12.5 \le t_3 < 16.5 \\ 4 & t_3 \ge 16.5 \end{cases}$$

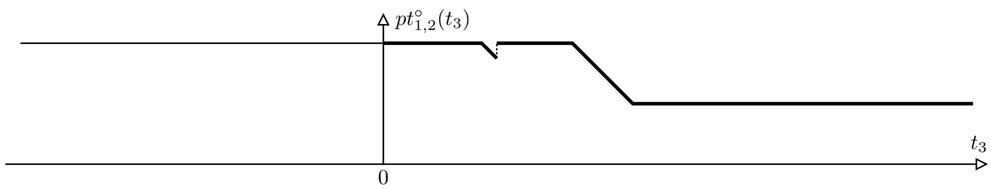


Figure 127: Optimal processing time $pt^\circ_{1,2}(t_3)$, under the assumption $\delta_1 = 1$ in state $[1\ 2\ 2\ t_3]^T$.

The conditioned cost-to-go $J^\circ_{1,2,2}(t_3 \mid \delta_1 = 1) = f(pt^\circ_{1,2}(t_3) + t_3) + g(pt^\circ_{1,2}(t_3))$, illustrated in figure 129, is provided by lemma 2. It is specified by the initial value 1.5, by the set { 6.5, 7.5, 9, 12.5, 19.5, 20.5, 22.5 } of abscissae γ_i, $i = 1, \ldots, 7$, at which the slope changes, and by the set { 1, 0, 0.5, 1, 1.5, 2.5, 3.5 } of slopes μ_i, $i = 1, \ldots, 7$, in the various intervals.

Case ii) in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{2,3}$, the following function

$$\alpha_{2,3}\ \max\{t_3 + st_{2,2} + \tau - dd_{2,3}\ ,\ 0\} + \beta_2\,(pt_2^{\mathrm{nom}} - \tau) + sc_{2,2} + J^\circ_{1,3,2}(t_4)$$

that can be written as $f(pt_{2,3} + t_3) + g(pt_{2,3})$ being

$$f(pt_{2,3} + t_3) = \max\{pt_{2,3} + t_3 - 38\ ,\ 0\} + J^\circ_{1,3,2}(pt_{2,3} + t_3)$$

$$g(pt_{2,3}) = \begin{cases} 1.5 \cdot (6 - pt_{2,3}) & pt_{2,3} \in [4, 6) \\ 0 & pt_{2,3} \notin [4, 6) \end{cases}$$

The function $pt^\circ_{2,3}(t_3) = \arg\min_{pt_{2,3}}\{f(pt_{2,3} + t_3) + g(pt_{2,3})\}$, with $4 \le pt_{2,3} \le 6$, is determined by applying lemma 1. It is (see figure 128)

$$pt^\circ_{2,3}(t_3) = x_{\mathrm{e}}(t_3) \qquad \text{with} \quad x_{\mathrm{e}}(t_3) = \begin{cases} 6 & t_3 < 13.5 \\ -t_3 + 19.5 & 13.5 \le t_3 < 15.5 \\ 4 & t_3 \ge 15.5 \end{cases}$$

The conditioned cost-to-go $J^\circ_{1,2,2}(t_3 \mid \delta_2 = 1) = f(pt^\circ_{2,3}(t_3) + t_3) + g(pt^\circ_{2,3}(t_3))$, illustrated in figure 129, is provided by lemma 2. It is specified by the initial value 1, by the set { 6.5, 13.5, 16.5, 20.5, 34 } of abscissae γ_i,

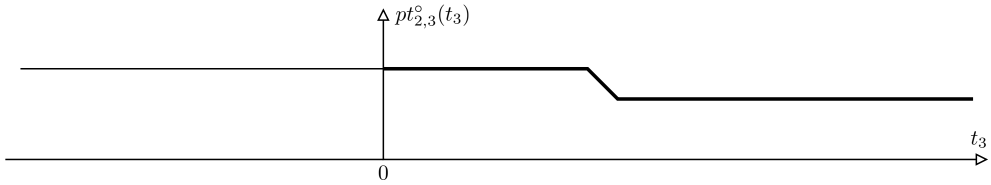


Figure 128: Optimal processing time $pt^\circ_{2,3}(t_3)$, under the assumption $\delta_2 = 1$ in state $[1\ 2\ 2\ t_3]^T$.

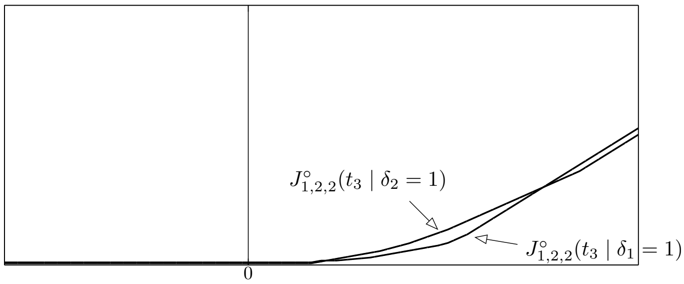


Figure 129: Conditioned costs-to-go $J^\circ_{1,2,2}(t_3 \mid \delta_1 = 1)$ and $J^\circ_{1,2,2}(t_3 \mid \delta_2 = 1)$ in state $[1\ 2\ 2\ t_3]^T$.

$i = 1, \ldots, 5$, at which the slope changes, and by the set { 1, 1.5, 2, 2.5, 3.5 } of slopes μ_i, $i = 1, \ldots, 5$, in the various intervals.

In order to find the optimal cost-to-go $J^\circ_{1,2,2}(t_3)$, it is necessary to carry out the following minimization

$$J^\circ_{1,2,2}(t_3) = \min \left\{ J^\circ_{1,2,2}(t_3 \mid \delta_1 = 1)\, ,\, J^\circ_{1,2,2}(t_3 \mid \delta_2 = 1) \right\}$$

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 130.

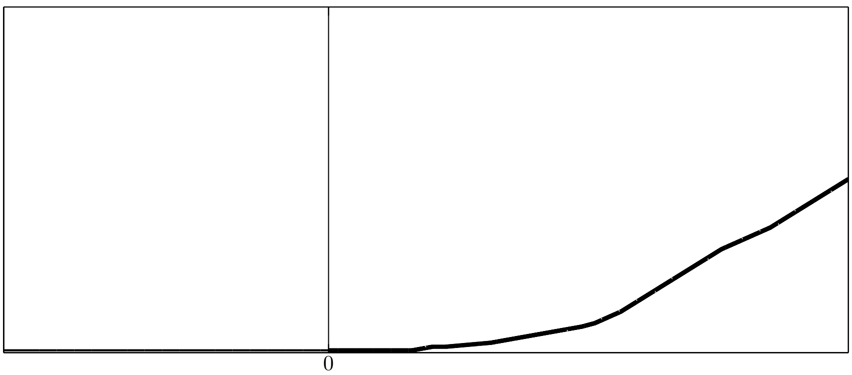


Figure 130: Optimal cost-to-go $J^\circ_{1,2,2}(t_3)$ in state $[1\ 2\ 2\ t_3]^T$.

The function $J^\circ_{1,2,2}(t_3)$ is specified by the initial value 1, by the set { 6.5, 8, 9, 12.5, 19.5, 20.5, 22.5, 30.25, 34 } of abscissae γ_i, $i = 1, \ldots, 9$, at which the slope changes, and by the set { 1, 0, 0.5, 1, 1.5, 2.5, 3.5, 2.5, 3.5 } of slopes μ_i, $i = 1, \ldots, 9$, in the various intervals.

Since $J^\circ_{1,2,2}(t_3 \mid \delta_1 = 1)$ is the minimum in $[8, 30.25)$, and $J^\circ_{1,2,2}(t_3 \mid \delta_2 = 1)$ is the minimum in $(-\infty, 8)$ and in $[30.25, +\infty)$, the optimal control strategies for this state are

$$\delta^\circ_1(1,2,2,t_3) = \begin{cases} 0 & t_3 < 8 \\ 1 & 8 \le t_3 < 30.25 \\ 0 & t_3 \ge 30.25 \end{cases} \qquad \delta^\circ_2(1,2,2,t_3) = \begin{cases} 1 & t_3 < 8 \\ 0 & 8 \le t_3 < 30.25 \\ 1 & t_3 \ge 30.25 \end{cases}$$

$$\tau^\circ(1,2,2,t_3) = \begin{cases} 6 & t_3 < 8 \\ 8 & 8 \le t_{34} < 12.5 \\ -t_3 + 20.5 & 12.5 \le t_3 < 16.5 \\ 4 & t_3 \ge 16.5 \end{cases}$$

The optimal control strategy $\tau^\circ(1,2,2,t_3)$ is illustrated in figure 131.

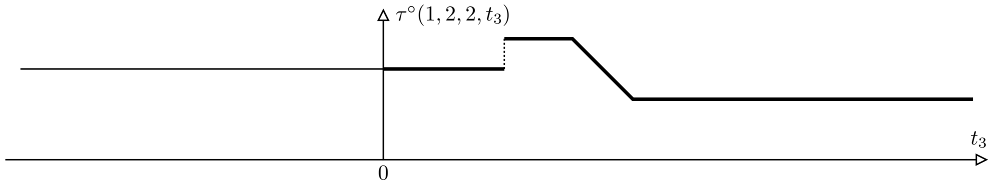


Figure 131: Optimal control strategy $\tau^\circ(1,2,2,t_3)$ in state $[1\ 2\ 2\ t_3]^T$.

Stage 3 – State $[\mathbf{1\ 2\ 1}\ \boldsymbol{t_3}]^{\boldsymbol{T}}$ ($S10$)

In state $[1\ 2\ 1\ t_3]^T$, the cost function to be minimized, with respect to the (continuos) decision variable τ and to the (binary) decision variables δ_1 and δ_2 is

$$\begin{aligned}&\delta_1\big[\alpha_{1,2}\ \max\{t_3 + st_{1,1} + \tau - dd_{1,2}\ ,\ 0\} + \beta_1\,(pt_1^{\text{nom}} - \tau) + sc_{1,1} + J^\circ_{2,2,1}(t_4)\big] + \\ &+ \delta_2\big[\alpha_{2,3}\ \max\{t_3 + st_{1,2} + \tau - dd_{2,3}\ ,\ 0\} + \beta_2\,(pt_2^{\text{nom}} - \tau) + sc_{1,2} + J^\circ_{1,3,2}(t_4)\big]\end{aligned}$$

Case i) in which it is assumed $\delta_1 = 1$ (and $\delta_2 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{1,2}$, the following function

$$\alpha_{1,2}\ \max\{t_3 + st_{1,1} + \tau - dd_{1,2}\ ,\ 0\} + \beta_1\,(pt_1^{\text{nom}} - \tau) + sc_{1,1} + J^\circ_{2,2,1}(t_4)$$

that can be written as $f(pt_{1,2} + t_3) + g(pt_{1,2})$ being

$$f(pt_{1,2} + t_3) = 0.5 \cdot \max\{pt_{1,2} + t_3 - 24\ ,\ 0\} + J^\circ_{2,2,1}(pt_{1,2} + t_3)$$

$$g(pt_{1,2}) = \begin{cases} 8 - pt_{1,2} & pt_{1,2} \in [4,8) \\ 0 & pt_{1,2} \notin [4,8) \end{cases}$$

The function $pt^\circ_{1,2}(t_3) = \arg\min_{pt_{1,2}}\{f(pt_{1,2} + t_3) + g(pt_{1,2})\}$, with $4 \le pt_{1,2} \le 8$, is determined by applying lemma 1. It is (see figure 132)

$$pt^\circ_{1,2}(t_3) = \begin{cases} x_{\text{s}}(t_3) & t_3 < 8 \\ x_{\text{e}}(t_3) & t_3 \ge 8 \end{cases} \qquad \text{with} \quad x_{\text{s}}(t_3) = \begin{cases} 8 & t_3 < 7 \\ -t_3 + 15 & 7 \le t_3 < 8 \end{cases}\ ,$$

$$\text{and} \quad x_{\text{e}}(t_3) = \begin{cases} 8 & 8 \le t_3 < 13 \\ -t_3 + 21 & 13 \le t_3 < 17 \\ 4 & t_3 \ge 17 \end{cases}$$

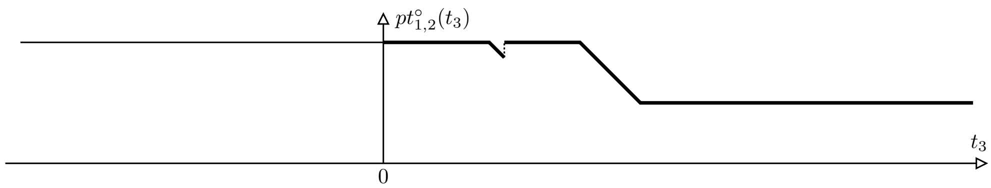


Figure 132: Optimal processing time $pt^\circ_{1,2}(t_3)$, under the assumption $\delta_1 = 1$ in state $[1\ 2\ 1\ t_3]^T$.

The conditioned cost-to-go $J^\circ_{1,2,1}(t_3 \mid \delta_1 = 1) = f(pt^\circ_{1,2}(t_3) + t_3) + g(pt^\circ_{1,2}(t_3))$, illustrated in figure 134, is provided by lemma 2. It is specified by the initial value 0.5, by the set { 7, 8, 9.5, 13, 20, 21, 23 } of abscissae γ_i, $i = 1, \dots, 7$, at which the slope changes, and by the set { 1, 0, 0.5, 1, 1.5, 2.5, 3.5 } of slopes μ_i, $i = 1, \dots, 7$, in the various intervals.

Case ii) in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{2,3}$, the following function

$$\alpha_{2,3}\ \max\{t_3 + st_{1,2} + \tau - dd_{2,3}\ ,\ 0\} + \beta_2\,(pt_2^{\text{nom}} - \tau) + sc_{1,2} + J^\circ_{1,3,2}(t_4)$$

that can be written as $f(pt_{2,3} + t_3) + g(pt_{2,3})$ being

$$f(pt_{2,3} + t_3) = \max\{pt_{2,3} + t_3 - 37\,,\,0\} + 0.5 + J^\circ_{1,3,2}(pt_{2,3} + t_3 + 1)$$

$$g(pt_{2,3}) = \begin{cases} 1.5 \cdot (6 - pt_{2,3}) & pt_{2,3} \in [4, 6) \\ 0 & pt_{2,3} \notin [4, 6) \end{cases}$$

The function $pt^\circ_{2,3}(t_3) = \arg\min_{pt_{2,3}}\{f(pt_{2,3} + t_3) + g(pt_{2,3})\}$, with $4 \le pt_{2,3} \le 6$, is determined by applying lemma 1. It is (see figure 133)

$$pt^\circ_{2,3}(t_3) = x_{\mathrm{e}}(t_3) \qquad \text{with} \quad x_{\mathrm{e}}(t_3) = \begin{cases} 6 & t_3 < 12.5 \\ -t_3 + 18.5 & 12.5 \le t_3 < 14.5 \\ 4 & t_3 \ge 14.5 \end{cases}$$

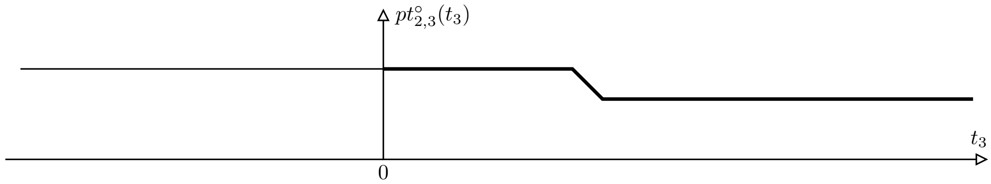


Figure 133: Optimal processing time $pt^\circ_{2,3}(t_3)$, under the assumption $\delta_2 = 1$ in state $[1\ 2\ 1\ t_3]^T$.

The conditioned cost-to-go $J^\circ_{1,2,1}(t_3 \mid \delta_2 = 1) = f(pt^\circ_{2,3}(t_3) + t_3) + g(pt^\circ_{2,3}(t_3))$, illustrated in figure 134, is provided by lemma 2. It is specified by the initial value 1.5, by the set { 5.5, 12.5, 15.5, 19.5, 33 } of abscissae γ_i, $i = 1, \ldots, 5$, at which the slope changes, and by the set { 1, 1.5, 2, 2.5, 3.5 } of slopes μ_i, $i = 1, \ldots, 5$, in the various intervals.

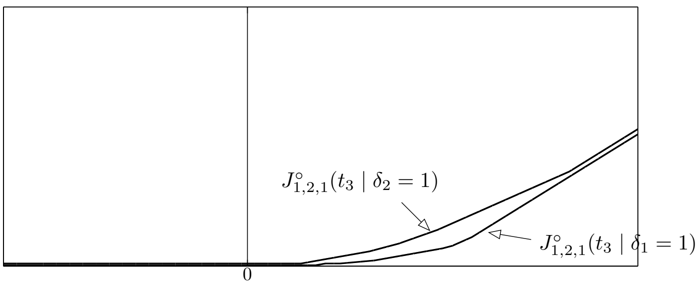


Figure 134: Conditioned costs-to-go $J^\circ_{1,2,1}(t_3 \mid \delta_1 = 1)$ and $J^\circ_{1,2,1}(t_3 \mid \delta_2 = 1)$ in state $[1\ 2\ 1\ t_3]^T$.

In order to find the optimal cost-to-go $J^\circ_{1,2,1}(t_3)$, it is necessary to carry out the following minimization

$$J^\circ_{1,2,1}(t_3) = \min\left\{J^\circ_{1,2,1}(t_3 \mid \delta_1 = 1)\,,\, J^\circ_{1,2,1}(t_3 \mid \delta_2 = 1)\right\}$$

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 135.

The function $J^\circ_{1,2,1}(t_3)$ is specified by the initial value 0.5, by the set { 7, 8, 9.5, 13, 20, 21, 23 } of abscissae γ_i, $i = 1, \ldots, 7$, at which the slope changes, and by the set { 1, 0, 0.5, 1, 1.5, 2.5, 3.5 } of slopes μ_i, $i = 1, \ldots, 7$, in the various intervals.

Since $J^\circ_{1,2,1}(t_3 \mid \delta_1 = 1)$ is always the minimum (see again figure 134), the optimal control strategies for this state are

$$\delta^\circ_1(1, 2, 1, t_3) = 1 \quad \forall\, t_3 \qquad \delta^\circ_2(1, 2, 1, t_3) = 0 \quad \forall\, t_3$$

$$\tau^\circ(1, 2, 1, t_3) = \begin{cases} 8 & t_3 < 7 \\ -t_3 + 15 & 7 \le t_3 < 8 \\ 8 & 8 \le t_3 < 13 \\ -t_3 + 21 & 13 \le t_3 < 17 \\ 4 & t_3 \ge 17 \end{cases}$$

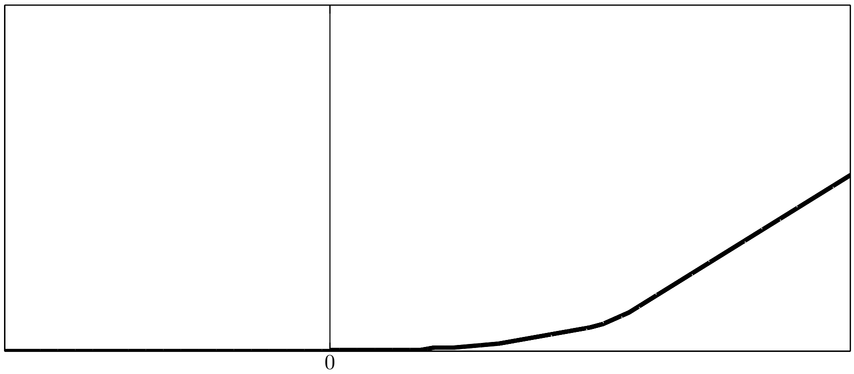


Figure 135: Optimal cost-to-go $J^{\circ}_{1,2,1}(t_3)$ in state $[1\ 2\ 1\ t_3]^T$.

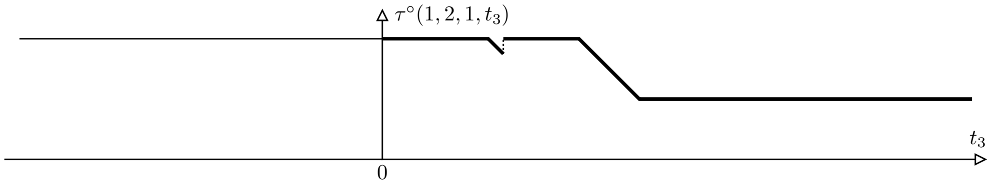


Figure 136: Optimal control strategy $\tau^{\circ}(1,2,1,t_3)$ in state $[1\ 2\ 1\ t_3]^T$.

The optimal control strategy $\tau^{\circ}(1,2,1,t_3)$ is illustrated in figure 136.

Stage 3 – State $[\mathbf{2\ 1\ 2}\ \boldsymbol{t_3}]^{\boldsymbol{T}}$ ($S9$)

In state $[2\ 1\ 2\ t_3]^T$, the cost function to be minimized, with respect to the (continuos) decision variable τ and to the (binary) decision variables δ_1 and δ_2 is

$$\delta_1\big[\alpha_{1,3}\ \max\{t_3 + st_{2,1} + \tau - dd_{1,3}\ ,\ 0\} + \beta_1\,(pt_1^{\mathrm{nom}} - \tau) + sc_{2,1} + J^{\circ}_{3,1,1}(t_4)\big] +$$
$$+\ \delta_2\big[\alpha_{2,2}\ \max\{t_3 + st_{2,2} + \tau - dd_{2,2}\ ,\ 0\} + \beta_2\,(pt_2^{\mathrm{nom}} - \tau) + sc_{2,2} + J^{\circ}_{2,2,2}(t_4)\big]$$

Case i) in which it is assumed $\delta_1 = 1$ (and $\delta_2 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{1,3}$, the following function

$$\alpha_{1,3}\ \max\{t_3 + st_{2,1} + \tau - dd_{1,3}\ ,\ 0\} + \beta_1\,(pt_1^{\mathrm{nom}} - \tau) + sc_{2,1} + J^{\circ}_{3,1,1}(t_4)$$

that can be written as $f(pt_{1,3} + t_3) + g(pt_{1,3})$ being

$$f(pt_{1,3} + t_3) = 1.5 \cdot \max\{pt_{1,3} + t_3 - 28.5\ ,\ 0\} + 1 + J^{\circ}_{3,1,1}(pt_{1,3} + t_3 + 0.5)$$

$$g(pt_{1,3}) = \begin{cases} 8 - pt_{1,3} & pt_{1,3} \in [4,8) \\ 0 & pt_{1,3} \notin [4,8) \end{cases}$$

The function $pt^{\circ}_{1,3}(t_3) = \arg\min_{pt_{1,3}}\{f(pt_{1,3} + t_3) + g(pt_{1,3})\}$, with $4 \le pt_{1,3} \le 8$, is determined by applying lemma 1. It is (see figure 137)

$$pt^{\circ}_{1,3}(t_3) = \begin{cases} x_{\mathrm{s}}(t_3) & t_3 < 1.5 \\ x_{\mathrm{e}}(t_3) & t_3 \ge 1.5 \end{cases} \qquad \text{with} \quad x_{\mathrm{s}}(t_3) = \begin{cases} 8 & t_3 < 0.5 \\ -t_3 + 8.5 & 0.5 \le t_3 < 1.5 \end{cases},$$

$$\text{and} \quad x_{\mathrm{e}}(t_3) = \begin{cases} 8 & 1.5 \le t_3 < 8.5 \\ -t_3 + 16.5 & 8.5 \le t_3 < 12.5 \\ 4 & t_3 \ge 12.5 \end{cases}$$

The conditioned cost-to-go $J^{\circ}_{2,1,2}(t_3 \mid \delta_1 = 1) = f(pt^{\circ}_{1,3}(t_3) + t_3) + g(pt^{\circ}_{1,3}(t_3))$, illustrated in figure 139, is provided by lemma 2. It is specified by the initial value 1.5, by the set { 0.5, 1.5, 8.5, 15, 22.5, 24.5 } of abscissae

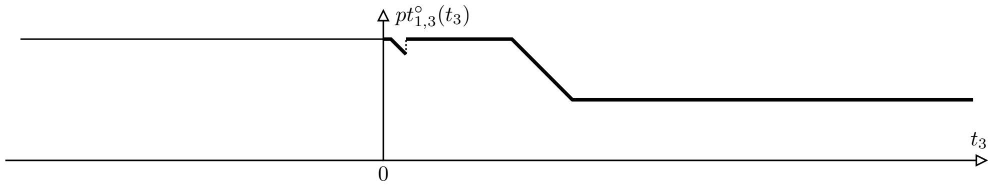


Figure 137: Optimal processing time $pt^{\circ}_{1,3}(t_3)$, under the assumption $\delta_1 = 1$ in state $[2\ 1\ 2\ t_3]^T$.

γ_i, $i = 1, \ldots, 6$, at which the slope changes, and by the set { 1, 0, 1, 1.5, 2.5, 4 } of slopes μ_i, $i = 1, \ldots, 6$, in the various intervals.

Case ii) in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{2,2}$, the following function

$$\alpha_{2,2}\ \max\{t_3 + st_{2,2} + \tau - dd_{2,2}\,,\ 0\} + \beta_2\,(pt_2^{\text{nom}} - \tau) + sc_{2,2} + J^{\circ}_{2,2,2}(t_4)$$

that can be written as $f(pt_{2,2} + t_3) + g(pt_{2,2})$ being

$$f(pt_{2,2} + t_3) = \max\{pt_{2,2} + t_3 - 24\,,\ 0\} + J^{\circ}_{2,2,2}(pt_{2,2} + t_3)$$

$$g(pt_{2,2}) = \begin{cases} 1.5 \cdot (6 - pt_{2,2}) & pt_{2,2} \in [4, 6) \\ 0 & pt_{2,2} \notin [4, 6) \end{cases}$$

The function $pt^{\circ}_{2,2}(t_3) = \arg\min_{pt_{2,2}} \{f(pt_{2,2} + t_3) + g(pt_{2,2})\}$, with $4 \leq pt_{2,2} \leq 6$, is determined by applying lemma 1. It is (see figure 138)

$$pt^{\circ}_{2,2}(t_3) = x_{\text{e}}(t_3) \qquad \text{with} \quad x_{\text{e}}(t_3) = \begin{cases} 6 & t_3 < 18 \\ -t_3 + 24 & 18 \leq t_3 < 20 \\ 4 & t_3 \geq 20 \end{cases}$$

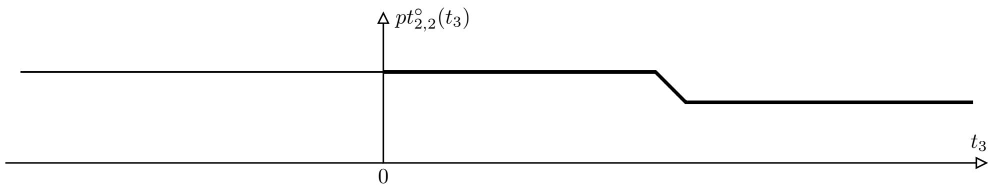


Figure 138: Optimal processing time $pt^{\circ}_{2,2}(t_3)$, under the assumption $\delta_2 = 1$ in state $[2\ 1\ 2\ t_3]^T$.

The conditioned cost-to-go $J^{\circ}_{2,1,2}(t_3 \mid \delta_2 = 1) = f(pt^{\circ}_{2,2}(t_3) + t_3) + g(pt^{\circ}_{2,2}(t_3))$, illustrated in figure 139, is provided by lemma 2. It is specified by the initial value 1, by the set { 8.5, 10, 11, 14.5, 18, 20, 20.5, 22.5, 28.25, 30 } of abscissae γ_i, $i = 1, \ldots, 10$, at which the slope changes, and by the set { 1, 0, 0.5, 1, 1.5, 2, 3, 4, 3, 4 } of slopes μ_i, $i = 1, \ldots, 10$, in the various intervals.

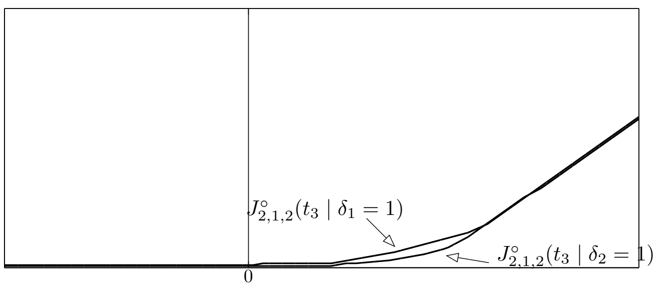


Figure 139: Conditioned costs-to-go $J^{\circ}_{2,1,2}(t_3 \mid \delta_1 = 1)$ and $J^{\circ}_{2,1,2}(t_3 \mid \delta_2 = 1)$ in state $[2\ 1\ 2\ t_3]^T$.

In order to find the optimal cost-to-go $J^{\circ}_{2,1,2}(t_3)$, it is necessary to carry out the following minimization

$$J^{\circ}_{2,1,2}(t_3) = \min\left\{J^{\circ}_{2,1,2}(t_3 \mid \delta_1 = 1)\,,\ J^{\circ}_{2,1,2}(t_3 \mid \delta_2 = 1)\right\}$$

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 140.

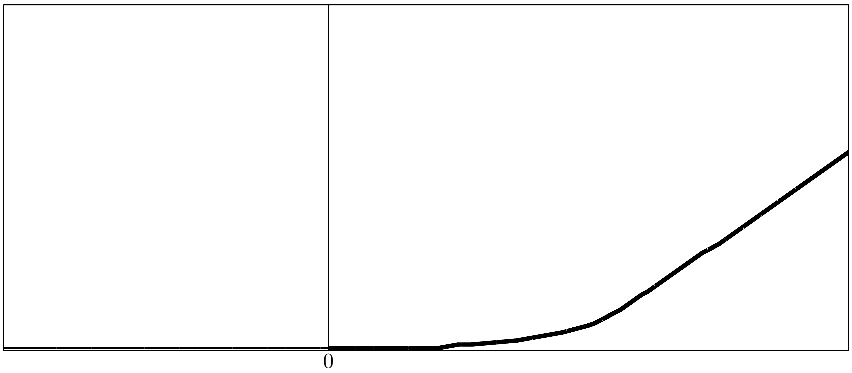


Figure 140: Optimal cost-to-go $J^{\circ}_{2,1,2}(t_3)$ in state $[2\ 1\ 2\ t_3]^T$.

The function $J^{\circ}_{2,1,2}(t_3)$ is specified by the initial value 1, by the set { 8.5, 10, 11, 14.5, 18, 20, 20.5, 22.5, 24,1$\overline{6}$, 24.5, 28.75, 30 } of abscissae γ_i, $i = 1, \ldots, 12$, at which the slope changes, and by the set { 1, 0, 0.5, 1, 1.5, 2, 3, 4, 2.5, 4, 3, 4 } of slopes μ_i, $i = 1, \ldots, 10$, in the various intervals.

Since $J^{\circ}_{2,1,2}(t_3 \mid \delta_1 = 1)$ is the minimum in $[24.1\overline{6}, 28.75)$, and $J^{\circ}_{2,1,2}(t_3 \mid \delta_2 = 1)$ is the minimum in $(-\infty, 24.1\overline{6})$ and in $[28.75, +\infty)$, the optimal control strategies for this state are

$$\delta^{\circ}_1(2,1,2,t_3) = \begin{cases} 0 & t_3 < 24.1\overline{6} \\ 1 & 24.1\overline{6} \le t_3 < 28.75 \\ 0 & t_3 \ge 28.75 \end{cases} \qquad \delta^{\circ}_2(2,1,2,t_3) = \begin{cases} 1 & t_3 < 24.1\overline{6} \\ 0 & 24.1\overline{6} \le t_3 < 28.75 \\ 1 & t_3 \ge 28.75 \end{cases}$$

$$\tau^{\circ}(2,1,2,t_3) = \begin{cases} 6 & t_3 < 18 \\ -t_3 + 24 & 18 \le t_3 < 20 \\ 4 & t_3 \ge 20 \end{cases}$$

The optimal control strategy $\tau^{\circ}(2,1,2,t_3)$ is illustrated in figure 141.

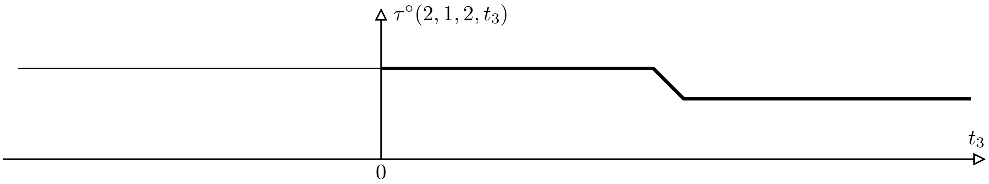


Figure 141: Optimal control strategy $\tau^{\circ}(2,1,2,t_3)$ in state $[2\ 1\ 2\ t_3]^T$.

Stage 3 – State $[\mathbf{2\ 1\ 1}\ \boldsymbol{t_3}]^{\boldsymbol{T}}$ ($S8$)

In state $[2\ 1\ 1\ t_3]^T$, the cost function to be minimized, with respect to the (continuos) decision variable τ and to the (binary) decision variables δ_1 and δ_2 is

$$\delta_1\big[\alpha_{1,3}\ \max\{t_3 + st_{1,1} + \tau - dd_{1,3}\,,\,0\} + \beta_1\,(pt_1^{\text{nom}} - \tau) + sc_{1,1} + J^{\circ}_{3,1,1}(t_4)\big] +$$
$$+\,\delta_2\big[\alpha_{2,2}\ \max\{t_3 + st_{1,2} + \tau - dd_{2,2}\,,\,0\} + \beta_2\,(pt_2^{\text{nom}} - \tau) + sc_{1,2} + J^{\circ}_{2,2,2}(t_4)\big]$$

Case i) in which it is assumed $\delta_1 = 1$ (and $\delta_2 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{1,3}$, the following function

$$\alpha_{1,3}\ \max\{t_3 + st_{1,1} + \tau - dd_{1,3}\,,\,0\} + \beta_1\,(pt_1^{\text{nom}} - \tau) + sc_{1,1} + J^{\circ}_{3,1,1}(t_4)$$

that can be written as $f(pt_{1,3} + t_3) + g(pt_{1,3})$ being

$$f(pt_{1,3} + t_3) = 1.5 \cdot \max\{pt_{1,3} + t_3 - 29\,,\,0\} + J^{\circ}_{3,1,1}(pt_{1,3} + t_3)$$

$$g(pt_{1,3}) = \begin{cases} 8 - pt_{1,3} & pt_{1,3} \in [4,8) \\ 0 & pt_{1,3} \notin [4,8) \end{cases}$$

The function $pt^{\circ}_{1,3}(t_3) = \arg\min_{pt_{1,3}} \{ f(pt_{1,3} + t_3) + g(pt_{1,3}) \}$, with $4 \le pt_{1,3} \le 8$, is determined by applying lemma 1. It is (see figure 142)

$$pt^{\circ}_{1,3}(t_3) = \begin{cases} x_{\mathrm{s}}(t_3) & t_3 < 2 \\ x_{\mathrm{e}}(t_3) & t_3 \ge 2 \end{cases} \qquad \text{with} \quad x_{\mathrm{s}}(t_3) = \begin{cases} 8 & t_3 < 1 \\ -t_3 + 9 & 1 \le t_3 < 2 \end{cases} ,$$

$$\text{and} \quad x_{\mathrm{e}}(t_3) = \begin{cases} 8 & 1.5 \le t_3 < 9 \\ -t_3 + 17 & 9 \le t_3 < 13 \\ 4 & t_3 \ge 13 \end{cases}$$

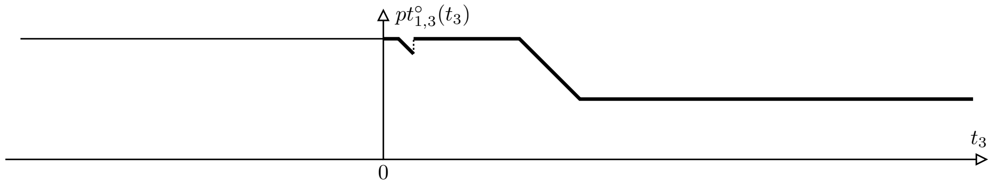


Figure 142: Optimal processing time $pt^{\circ}_{1,3}(t_3)$, under the assumption $\delta_1 = 1$ in state $[2\ 1\ 1\ t_3]^T$.

The conditioned cost-to-go $J^{\circ}_{2,1,1}(t_3 \mid \delta_1 = 1) = f(pt^{\circ}_{1,3}(t_3) + t_3) + g(pt^{\circ}_{1,3}(t_3))$, illustrated in figure 144, is provided by lemma 2. It is specified by the initial value 0.5, by the set { 1, 2, 9, 15.5, 23, 25 } of abscissae γ_i, $i = 1, \ldots, 6$, at which the slope changes, and by the set { 1, 0, 1, 1.5, 2.5, 4 } of slopes μ_i, $i = 1, \ldots, 6$, in the various intervals.

Case ii) in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{2,2}$, the following function

$$\alpha_{2,2} \, \max\{t_3 + st_{1,2} + \tau - dd_{2,2} \, , \, 0\} + \beta_2 \, (pt_2^{\mathrm{nom}} - \tau) + sc_{1,2} + J^{\circ}_{2,2,2}(t_4)$$

that can be written as $f(pt_{2,2} + t_3) + g(pt_{2,2})$ being

$$f(pt_{2,2} + t_3) = \max\{pt_{2,2} + t_3 - 23 \, , \, 0\} + 0.5 + J^{\circ}_{2,2,2}(pt_{2,2} + t_3 + 1)$$

$$g(pt_{2,2}) = \begin{cases} 1.5 \cdot (6 - pt_{2,2}) & pt_{2,2} \in [4,6) \\ 0 & pt_{2,2} \notin [4,6) \end{cases}$$

The function $pt^{\circ}_{2,2}(t_3) = \arg\min_{pt_{2,2}} \{ f(pt_{2,2} + t_3) + g(pt_{2,2}) \}$, with $4 \le pt_{2,2} \le 6$, is determined by applying lemma 1. It is (see figure 143)

$$pt^{\circ}_{2,2}(t_3) = x_{\mathrm{e}}(t_3) \qquad \text{with} \quad x_{\mathrm{e}}(t_3) = \begin{cases} 6 & t_3 < 17 \\ -t_3 + 23 & 17 \le t_3 < 19 \\ 4 & t_3 \ge 19 \end{cases}$$

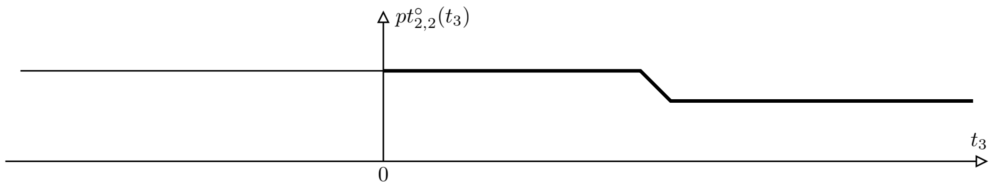


Figure 143: Optimal processing time $pt^{\circ}_{2,2}(t_3)$, under the assumption $\delta_2 = 1$ in state $[2\ 1\ 1\ t_3]^T$.

The conditioned cost-to-go $J^{\circ}_{2,1,1}(t_3 \mid \delta_2 = 1) = f(pt^{\circ}_{2,2}(t_3) + t_3) + g(pt^{\circ}_{2,2}(t_3))$, illustrated in figure 144, is provided by lemma 2. It is specified by the initial value 1.5, by the set { 7.5, 9, 10, 13.5, 17, 19, 19.5, 21.5, 27.25, 29 } of abscissae γ_i, $i = 1, \ldots, 10$, at which the slope changes, and by the set { 1, 0, 0.5, 1, 1.5, 2, 3, 4, 3, 4 } of slopes μ_i, $i = 1, \ldots, 10$, in the various intervals.

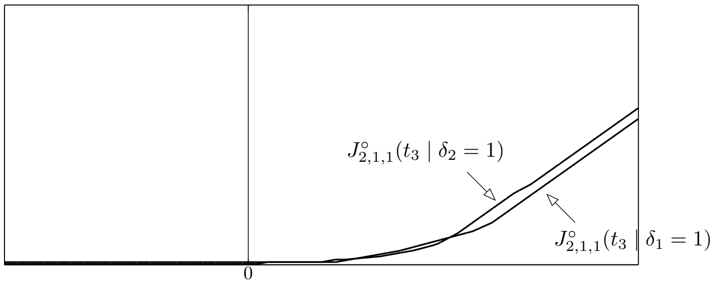


Figure 144: Conditioned costs-to-go $J^\circ_{2,1,1}(t_3 \mid \delta_1 = 1)$ and $J^\circ_{2,1,1}(t_3 \mid \delta_2 = 1)$ in state $[2\ 1\ 1\ t_3]^T$.

In order to find the optimal cost-to-go $J^\circ_{2,1,1}(t_3)$, it is necessary to carry out the following minimization

$$J^\circ_{2,1,1}(t_3) = \min \left\{ J^\circ_{2,1,1}(t_3 \mid \delta_1 = 1)\, , \, J^\circ_{2,1,1}(t_3 \mid \delta_2 = 1) \right\}$$

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 145.

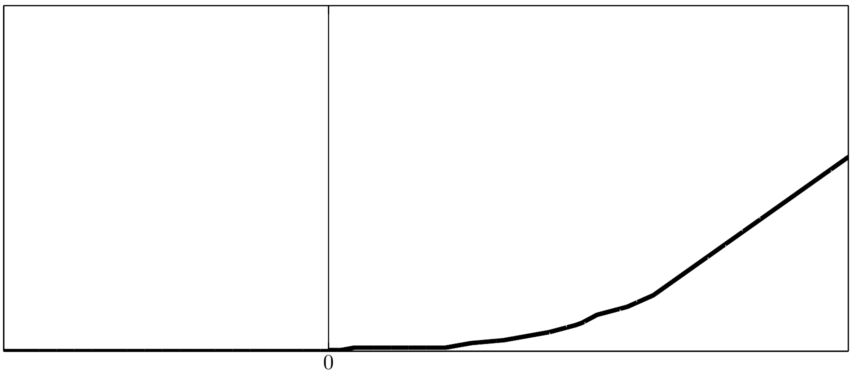


Figure 145: Optimal cost-to-go $J^\circ_{2,1,1}(t_3)$ in state $[2\ 1\ 1\ t_3]^T$.

The function $J^\circ_{2,1,1}(t_3)$ is specified by the initial value 0.5, by the set { 1, 2, 9, 11, 13.5, 17, 19, 19.5, 20.$\overline{6}$, 23, 25 } of abscissae γ_i, $i = 1, \ldots, 11$, at which the slope changes, and by the set { 1, 0, 1, 0.5, 1, 1.5, 2, 3, 1.5, 2.5, 4 } of slopes μ_i, $i = 1, \ldots, 11$, in the various intervals.

Since $J^\circ_{2,1,1}(t_3 \mid \delta_1 = 1)$ is the minimum in $(-\infty, 2)$, in $[9, 11]$, and in $[20.\overline{6}, +\infty)$, and $J^\circ_{2,1,1}(t_3 \mid \delta_2 = 1)$ is the minimum in $[2, 9]$ and in $[11, 20.\overline{6}]$, the optimal control strategies for this state are

$$\delta^\circ_1(2,1,1,t_3) = \begin{cases} 1 & t_3 < 2 \\ 0 & 2 \le t_3 < 9 \\ 1 & 9 \le t_3 < 11 \\ 0 & 11 \le t_3 < 20.\overline{6} \\ 1 & t_3 \ge 20.\overline{6} \end{cases} \qquad \delta^\circ_2(2,1,1,t_3) = \begin{cases} 0 & t_3 < 2 \\ 1 & 2 \le t_3 < 9 \\ 0 & 9 \le t_3 < 11 \\ 1 & 11 \le t_3 < 20.\overline{6} \\ 0 & t_3 \ge 20.\overline{6} \end{cases}$$

$$\tau^\circ(2,1,1,t_3) = \begin{cases} 8 & t_3 < 1 \\ -t_3 + 9 & 1 \le t_3 < 2 \\ 6 & 2 \le t_3 < 9 \\ -t_3 + 17 & 9 \le t_3 < 11 \\ 6 & 11 \le t_3 < 17 \\ -t_3 + 23 & 17 \le t_3 < 19 \\ 4 & t_3 \ge 19 \end{cases}$$

The optimal control strategy $\tau^\circ(2,1,1,t_3)$ is illustrated in figure 146.

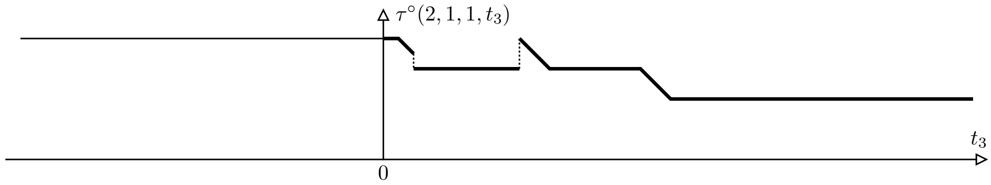


Figure 146: Optimal control strategy $\tau^{\circ}(2,1,1,t_3)$ in state $[2\ 1\ 1\ t_3]^T$.

Stage 3 – State $[\mathbf{3\ 0\ 1}\ \boldsymbol{t_3}]^{\boldsymbol{T}}$ ($S7$)

In state $[3\ 0\ 1\ t_3]^T$, the cost function to be minimized, with respect to the (continuos) decision variable τ and to the (binary) decision variables δ_1 and δ_2 is

$$\delta_1\big[\alpha_{1,4}\ \max\{t_3+st_{1,1}+\tau-dd_{1,4}\ ,\ 0\}+\beta_1\,(pt_1^{\mathrm{nom}}-\tau)+sc_{1,1}+J^{\circ}_{4,0,1}(t_4)\big]+$$
$$+\ \delta_2\big[\alpha_{2,1}\ \max\{t_3+st_{1,2}+\tau-dd_{2,1}\ ,\ 0\}+\beta_2\,(pt_2^{\mathrm{nom}}-\tau)+sc_{1,2}+J^{\circ}_{3,1,2}(t_4)\big]$$

Case i) in which it is assumed $\delta_1=1$ (and $\delta_2=0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{1,4}$, the following function

$$\alpha_{1,4}\ \max\{t_3+st_{1,1}+\tau-dd_{1,4}\ ,\ 0\}+\beta_1\,(pt_1^{\mathrm{nom}}-\tau)+sc_{1,1}+J^{\circ}_{4,0,1}(t_4)$$

that can be written as $f(pt_{1,4}+t_3)+g(pt_{1,4})$ being

$$f(pt_{1,4}+t_3)=0.5\cdot\max\{pt_{1,4}+t_3-41\ ,\ 0\}+J^{\circ}_{4,0,1}(pt_{1,4}+t_3)$$

$$g(pt_{1,4})=\begin{cases}8-pt_{1,4} & pt_{1,4}\in[4,8)\\ 0 & pt_{1,4}\notin[4,8)\end{cases}$$

The function $pt^{\circ}_{1,4}(t_3)=\arg\min_{pt_{1,4}}\{f(pt_{1,4}+t_3)+g(pt_{1,4})\}$, with $4\le pt_{1,4}\le 8$, is determined by applying lemma 1. It is (see figure 147)

$$pt^{\circ}_{1,4}(t_3)=x_{\mathrm{e}}(t_3)\qquad\text{with}\quad x_{\mathrm{e}}(t_3)=\begin{cases}8 & t_3<3\\ -t_3+23 & 3\le t_3<7\\ 4 & t_3\ge 7\end{cases}$$

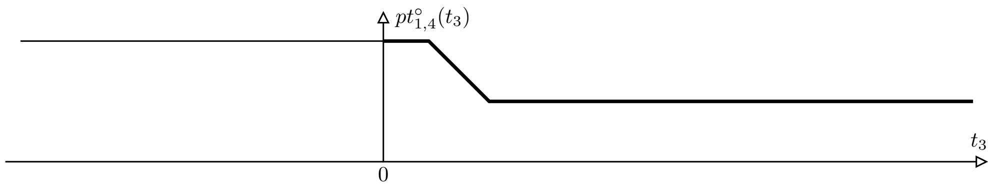


Figure 147: Optimal processing time $pt^{\circ}_{1,4}(t_3)$, under the assumption $\delta_1=1$ in state $[3\ 0\ 1\ t_3]^T$.

The conditioned cost-to-go $J^{\circ}_{3,0,1}(t_3\mid\delta_1=1)=f(pt^{\circ}_{1,4}(t_3)+t_3)+g(pt^{\circ}_{1,4}(t_3))$, illustrated in figure 149, is provided by lemma 2. It is specified by the initial value 0.5, by the set { 3, 10, 12, 17, 19, 37 } of abscissae γ_i, $i=1,\ldots,6$, at which the slope changes, and by the set { 1, 1.5, 3, 3.5, 4, 4.5 } of slopes μ_i, $i=1,\ldots,6$, in the various intervals.

Case ii) in which it is assumed $\delta_2=1$ (and $\delta_1=0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{2,1}$, the following function

$$\alpha_{2,1}\ \max\{t_3+st_{1,2}+\tau-dd_{2,1}\ ,\ 0\}+\beta_2\,(pt_2^{\mathrm{nom}}-\tau)+sc_{1,2}+J^{\circ}_{3,1,2}(t_4)$$

that can be written as $f(pt_{2,1}+t_3)+g(pt_{2,1})$ being

$$f(pt_{2,1}+t_3)=2\cdot\max\{pt_{2,1}+t_3-20\ ,\ 0\}+0.5+J^{\circ}_{3,1,2}(pt_{2,1}+t_3+1)$$

$$g(pt_{2,1}) = \begin{cases} 1.5 \cdot (6 - pt_{2,1}) & pt_{2,1} \in [4, 6) \\ 0 & pt_{2,1} \notin [4, 6) \end{cases}$$

The function $pt^{\circ}_{2,1}(t_3) = \arg\min_{pt_{2,1}} \{f(pt_{2,1} + t_3) + g(pt_{2,1})\}$, with $4 \leq pt_{2,1} \leq 6$, is determined by applying lemma 1. It is (see figure 148)

$$pt^{\circ}_{2,1}(t_3) = x_{\mathrm{e}}(t_3) \qquad \text{with} \quad x_{\mathrm{e}}(t_3) = \begin{cases} 6 & t_3 < 13.5 \\ -t_3 + 19.5 & 13.5 \leq t_3 < 15.5 \\ 4 & t_3 \geq 15.5 \end{cases}$$

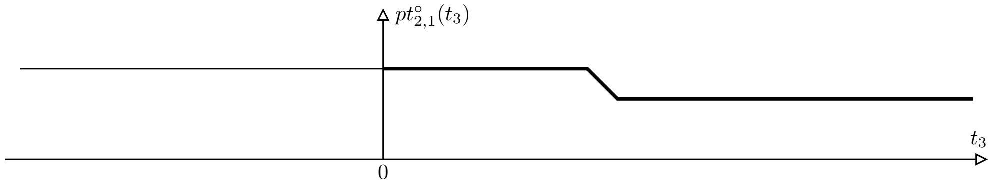


Figure 148: Optimal processing time $pt^{\circ}_{2,1}(t_3)$, under the assumption $\delta_2 = 1$ in state $[3\ 0\ 1\ t_3]^T$.

The conditioned cost-to-go $J^{\circ}_{3,0,1}(t_3 \mid \delta_2 = 1) = f(pt^{\circ}_{2,1}(t_3) + t_3) + g(pt^{\circ}_{2,1}(t_3))$, illustrated in figure 149, is provided by lemma 2. It is specified by the initial value 1.5, by the set { 11, 13.5, 16, 23 } of abscissae γ_i, $i = 1, \ldots, 4$, at which the slope changes, and by the set { 1, 1.5, 3.5, 4.5 } of slopes μ_i, $i = 1, \ldots, 4$, in the various intervals.

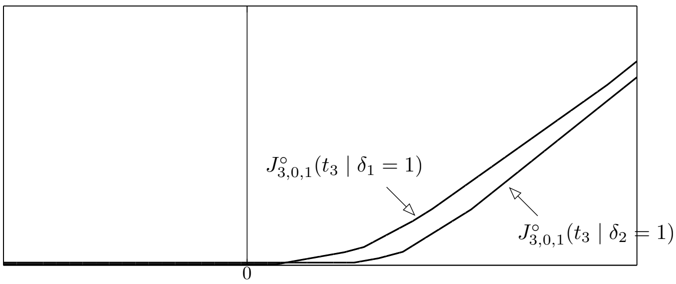


Figure 149: Conditioned costs-to-go $J^{\circ}_{3,0,1}(t_3 \mid \delta_1 = 1)$ and $J^{\circ}_{3,0,1}(t_3 \mid \delta_2 = 1)$ in state $[3\ 0\ 1\ t_3]^T$.

In order to find the optimal cost-to-go $J^{\circ}_{3,0,1}(t_3)$, it is necessary to carry out the following minimization

$$J^{\circ}_{3,0,1}(t_3) = \min \left\{ J^{\circ}_{3,0,1}(t_3 \mid \delta_1 = 1) \,,\, J^{\circ}_{3,0,1}(t_3 \mid \delta_2 = 1) \right\}$$

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 150.

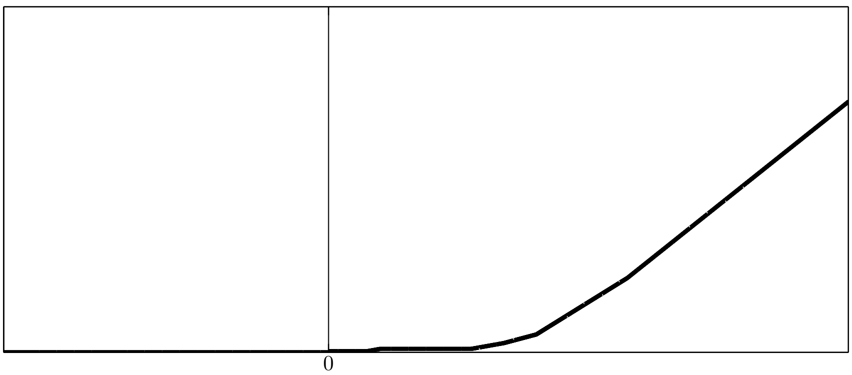


Figure 150: Optimal cost-to-go $J^{\circ}_{3,0,1}(t_3)$ in state $[3\ 0\ 1\ t_3]^T$.

The function $J^{\circ}_{3,0,1}(t_3)$ is specified by the initial value 0.5, by the set { 3, 4, 11, 13.5, 16, 23 } of abscissae γ_i, $i = 1, \ldots, 6$, at which the slope changes, and by the set { 1, 0, 1, 1.5, 3.5, 4.5 } of slopes μ_i, $i = 1, \ldots, 6$, in the various intervals.

Since $J^{\circ}_{3,0,1}(t_3 \mid \delta_1 = 1)$ is the minimum in $(-\infty, 4)$, and $J^{\circ}_{3,0,1}(t_3 \mid \delta_2 = 1)$ is the minimum in $[4, +\infty$, the optimal control strategies for this state are

$$\delta_1^{\circ}(3,0,1,t_3) = \begin{cases} 1 & t_3 < 4 \\ 0 & t_3 \geq 4 \end{cases} \qquad \delta_2^{\circ}(3,0,1,t_3) = \begin{cases} 0 & t_3 < 4 \\ 1 & t_3 \geq 4 \end{cases}$$

$$\tau^{\circ}(3,0,1,t_3) = \begin{cases} 8 & t_3 < 3 \\ -t_3 + 23 & 3 \leq t_3 < 4 \\ 6 & 4 \leq t_3 < 13.5 \\ -t_3 + 19.5 & 13.5 \leq t_3 < 15.5 \\ 4 & t_3 \geq 15.5 \end{cases}$$

The optimal control strategy $\tau^{\circ}(3,0,1,t_3)$ is illustrated in figure 146.

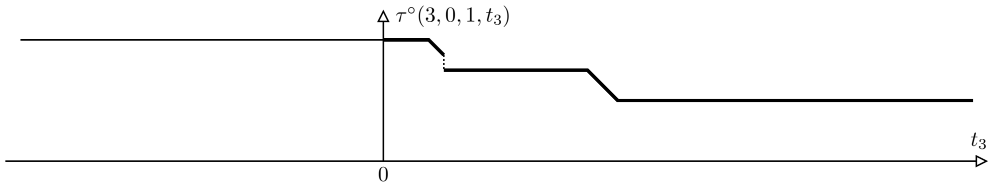


Figure 151: Optimal control strategy $\tau^{\circ}(3,0,1,t_3)$ in state $[3\ 0\ 1\ t_3]^T$.

Stage 2 – State $[\mathbf{0\ 2\ 2}\ \boldsymbol{t_2}]^{\boldsymbol{T}}$ ($S6$)

In state $[0\ 2\ 2\ t_2]^T$, the cost function to be minimized, with respect to the (continuos) decision variable τ and to the (binary) decision variables δ_1 and δ_2 is

$$\delta_1\big[\alpha_{1,1}\ \max\{t_2 + st_{2,1} + \tau - dd_{1,1}\ ,\ 0\} + \beta_1\,(pt_1^{\text{nom}} - \tau) + sc_{2,1} + J^{\circ}_{1,2,1}(t_3)\big] +$$
$$+\ \delta_2\big[\alpha_{2,3}\ \max\{t_2 + st_{2,2} + \tau - dd_{2,3}\ ,\ 0\} + \beta_2\,(pt_2^{\text{nom}} - \tau) + sc_{2,2} + J^{\circ}_{0,3,2}(t_3)\big]$$

Case i) in which it is assumed $\delta_1 = 1$ (and $\delta_2 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{1,1}$, the following function

$$\alpha_{1,1}\ \max\{t_2 + st_{2,1} + \tau - dd_{1,1}\ ,\ 0\} + \beta_1\,(pt_1^{\text{nom}} - \tau) + sc_{2,1} + J^{\circ}_{1,2,1}(t_3)$$

that can be written as $f(pt_{1,1} + t_2) + g(pt_{1,1})$ being

$$f(pt_{1,1} + t_2) = 0.75 \cdot \max\{pt_{1,1} + t_2 - 18.5\ ,\ 0\} + 1 + J^{\circ}_{1,2,1}(pt_{1,1} + t_2 + 0.5)$$

$$g(pt_{1,1}) = \begin{cases} 8 - pt_{1,1} & pt_{1,1} \in [4, 8) \\ 0 & pt_{1,1} \notin [4, 8) \end{cases}$$

The function $pt^{\circ}_{1,1}(t_2) = \arg\min_{pt_{1,1}}\{f(pt_{1,1} + t_2) + g(pt_{1,1})\}$, with $4 \leq pt_{1,1} \leq 8$, is determined by applying lemma 1. It is (see figure 152)

$$pt^{\circ}_{1,1}(t_2) = \begin{cases} x_{\text{s}}(t_2) & t_2 < -0.5 \\ x_{\text{e}}(t_2) & t_2 \geq -0.5 \end{cases} \qquad \text{with} \quad x_{\text{s}}(t_2) = \begin{cases} 8 & t_2 < -1.5 \\ -t_2 + 6.5 & -1.5 \leq t_2 < -0.5 \end{cases},$$

$$\text{and} \quad x_{\text{e}}(t_2) = \begin{cases} 8 & -0.5 \leq t_2 < 4.5 \\ -t_2 + 12.5 & 4.5 \leq t_2 < 8.5 \\ 4 & t_2 \geq 8.5 \end{cases}$$

The conditioned cost-to-go $J^{\circ}_{0,2,2}(t_2 \mid \delta_1 = 1) = f(pt^{\circ}_{1,1}(t_2) + t_2) + g(pt^{\circ}_{1,1}(t_2))$, illustrated in figure 154, is provided by lemma 2. It is specified by the initial value 1.5, by the set { –1.5, –0.5, 1, 4.5, 14.5, 15.5, 16.5, 18.5 } of abscissae γ_i, $i = 1, \ldots, 8$, at which the slope changes, and by the set { 1, 0, 0.5, 1, 1.75, 2.25, 3.25, 4.25 } of slopes μ_i, $i = 1, \ldots, 8$, in the various intervals.

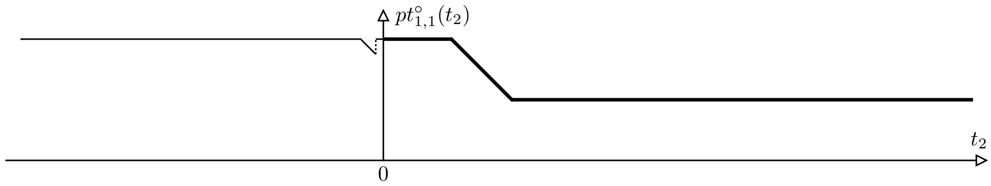


Figure 152: Optimal processing time $pt^\circ_{1,1}(t_2)$, under the assumption $\delta_1 = 1$ in state $[0\ 2\ 2\ t_2]^T$.

Case ii) in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{2,3}$, the following function

$$\alpha_{2,3}\ \max\{t_2 + st_{2,2} + \tau - dd_{2,3}\ ,\ 0\} + \beta_2\,(pt_2^{\text{nom}} - \tau) + sc_{2,2} + J^\circ_{0,3,2}(t_3)$$

that can be written as $f(pt_{2,3} + t_2) + g(pt_{2,3})$ being

$$f(pt_{2,3} + t_2) = \max\{pt_{2,3} + t_2 - 38\ ,\ 0\} + J^\circ_{0,3,2}(pt_{2,3} + t_2)$$

$$g(pt_{2,3}) = \begin{cases} 1.5 \cdot (6 - pt_{2,3}) & pt_{2,3} \in [4, 6) \\ 0 & pt_{2,3} \notin [4, 6) \end{cases}$$

The function $pt^\circ_{2,3}(t_2) = \arg\min_{pt_{2,3}}\{f(pt_{2,3} + t_2) + g(pt_{2,3})\}$, with $4 \le pt_{2,3} \le 6$, is determined by applying lemma 1. It is (see figure 153)

$$pt^\circ_{2,3}(t_2) = x_\text{e}(t_2) \qquad \text{with} \quad x_\text{e}(t_2) = \begin{cases} 6 & t_2 < 8.5 \\ -t_2 + 14.5 & 8.5 \le t_2 < 10.5 \\ 4 & t_2 \ge 10.5 \end{cases}$$

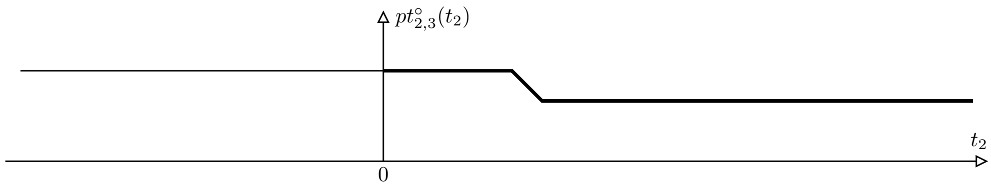


Figure 153: Optimal processing time $pt^\circ_{2,3}(t_2)$, under the assumption $\delta_2 = 1$ in state $[0\ 2\ 2\ t_2]^T$.

The conditioned cost-to-go $J^\circ_{0,2,2}(t_2 \mid \delta_2 = 1) = f(pt^\circ_{2,3}(t_2) + t_2) + g(pt^\circ_{2,3}(t_2))$, illustrated in figure 154, is provided by lemma 2. It is specified by the initial value 1, by the set { –1.5, 8.5, 10.5, 11.5, 12.5, 16.5, 34 } of abscissae γ_i, $i = 1, \ldots, 7$, at which the slope changes, and by the set { 1, 1.5, 1.75, 2.25, 2.75, 3.25, 4.25 } of slopes μ_i, $i = 1, \ldots, 7$, in the various intervals.

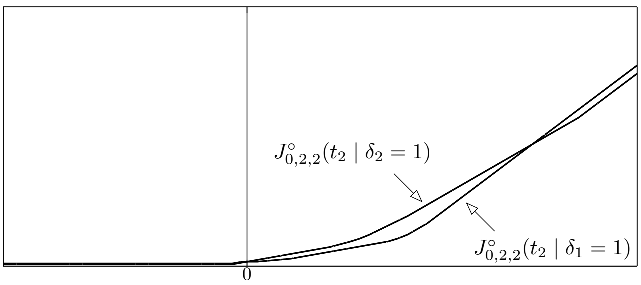


Figure 154: Conditioned costs-to-go $J^\circ_{0,2,2}(t_2 \mid \delta_1 = 1)$ and $J^\circ_{0,2,2}(t_2 \mid \delta_2 = 1)$ in state $[0\ 2\ 2\ t_2]^T$.

In order to find the optimal cost-to-go $J^\circ_{0,2,2}(t_2)$, it is necessary to carry out the following minimization

$$J^\circ_{0,2,2}(t_2) = \min\left\{J^\circ_{0,2,2}(t_2 \mid \delta_1 = 1)\ ,\ J^\circ_{0,2,2}(t_2 \mid \delta_2 = 1)\right\}$$

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 155.

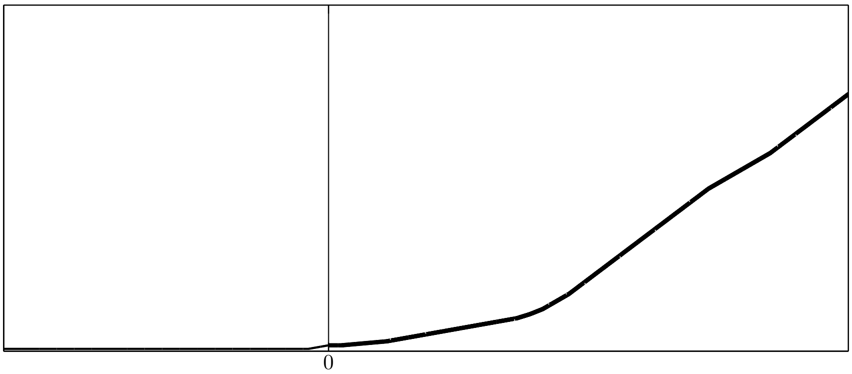


Figure 155: Optimal cost-to-go $J^{\circ}_{0,2,2}(t_2)$ in state $[0\ 2\ 2\ t_2]^T$.

The function $J^{\circ}_{0,2,2}(t_2)$ is specified by the initial value 1, by the set { –1.5, 0, 1, 4.5, 14.5, 15.5, 16.5, 18.5, 29.25, 34 } of abscissae γ_i, $i = 1, \dots, 10$, at which the slope changes, and by the set { 1, 0, 0.5, 1, 1.75, 2.25, 3.25, 4.25, 3.25, 4.25 } of slopes μ_i, $i = 1, \dots, 10$, in the various intervals.

Since $J^{\circ}_{0,2,2}(t_2 \mid \delta_1 = 1)$ is the minimum in $[0, 29.25)$, and $J^{\circ}_{0,2,2}(t_2 \mid \delta_2 = 1)$ is the minimum in $(-\infty, 0)$ and in $[29.25, +\infty)$, the optimal control strategies for this state are

$$\delta_1^{\circ}(0,2,2,t_2) = \begin{cases} 0 & t_2 < 0 \\ 1 & 0 \le t_2 < 29.25 \\ 0 & t_2 \ge 29.25 \end{cases} \qquad \delta_2^{\circ}(0,2,2,t_2) = \begin{cases} 1 & t_2 < 0 \\ 0 & 0 \le t_2 < 29.25 \\ 1 & t_2 \ge 29.25 \end{cases}$$

$$\tau^{\circ}(0,2,2,t_2) = \begin{cases} 6 & t_2 < 0 \\ 8 & 0 \le t_2 < 4.5 \\ -t_2 + 12.5 & 4.5 \le t_2 < 8.5 \\ 4 & t_2 \ge 8.5 \end{cases}$$

The optimal control strategy $\tau^{\circ}(0,2,2,t_2)$ is illustrated in figure 156.

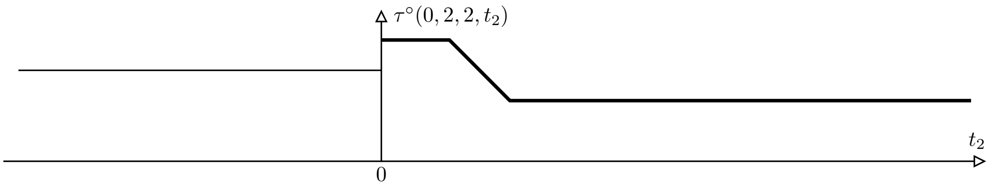


Figure 156: Optimal control strategy $\tau^{\circ}(0,2,2,t_2)$ in state $[0\ 2\ 2\ t_2]^T$.

Stage 2 – State $[\mathbf{1\ 1\ 2}\ \boldsymbol{t_2}]^{\boldsymbol{T}}$ ($S5$)

In state $[1\ 1\ 2\ t_2]^T$, the cost function to be minimized, with respect to the (continuos) decision variable τ and to the (binary) decision variables δ_1 and δ_2 is

$$\delta_1 \big[\alpha_{1,2}\, \max\{t_2 + st_{2,1} + \tau - dd_{1,2}\,,\, 0\} + \beta_1\,(pt_1^{\text{nom}} - \tau) + sc_{2,1} + J^{\circ}_{2,1,1}(t_3)\big] +$$
$$+\, \delta_2 \big[\alpha_{2,2}\, \max\{t_2 + st_{2,2} + \tau - dd_{2,2}\,,\, 0\} + \beta_2\,(pt_2^{\text{nom}} - \tau) + sc_{2,2} + J^{\circ}_{1,2,2}(t_3)\big]$$

Case i) in which it is assumed $\delta_1 = 1$ (and $\delta_2 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{1,2}$, the following function

$$\alpha_{1,2}\, \max\{t_2 + st_{2,1} + \tau - dd_{1,2}\,,\, 0\} + \beta_1\,(pt_1^{\text{nom}} - \tau) + sc_{2,1} + J^{\circ}_{2,1,1}(t_3)$$

that can be written as $f(pt_{1,2} + t_2) + g(pt_{1,2})$ being

$$f(pt_{1,2} + t_2) = 0.5 \cdot \max\{pt_{1,2} + t_2 - 23.5\,,\, 0\} + 1 + J^{\circ}_{2,1,1}(pt_{1,2} + t_2 + 0.5)$$

$$g(pt_{1,2}) = \begin{cases} 8 - pt_{1,2} & pt_{1,2} \in [4, 8) \\ 0 & pt_{1,2} \notin [4, 8) \end{cases}$$

The function $pt^{\circ}_{1,2}(t_2) = \arg\min_{pt_{1,2}}\{f(pt_{1,2} + t_2) + g(pt_{1,2})\}$, with $4 \leq pt_{1,2} \leq 8$, is determined by applying lemma 1. It is (see figure 157)

$$pt^{\circ}_{1,2}(t_2) = \begin{cases} x_{\rm s}(t_2) & t_2 < -6.5 \\ x_1(t_2) & -6.5 \leq t_2 < 2.5 \\ x_{\rm e}(t_2) & t_2 \geq 2.5 \end{cases} \qquad \text{with} \quad x_{\rm s}(t_2) = \begin{cases} 8 & t_2 < -7.5 \\ -t_2 + 0.5 & -7.5 \leq t_2 < -6.5 \end{cases},$$

$$x_1(t_2) = \begin{cases} 8 & -6.5 \leq t_2 < 0.5 \\ -t_2 + 8.5 & 0.5 \leq t_2 < 2.5 \end{cases}, \text{ and } \quad x_{\rm e}(t_2) = \begin{cases} 8 & 2.5 \leq t_2 < 5 \\ -t_2 + 13 & 5 \leq t_2 < 9 \\ 4 & t_2 \geq 9 \end{cases}$$

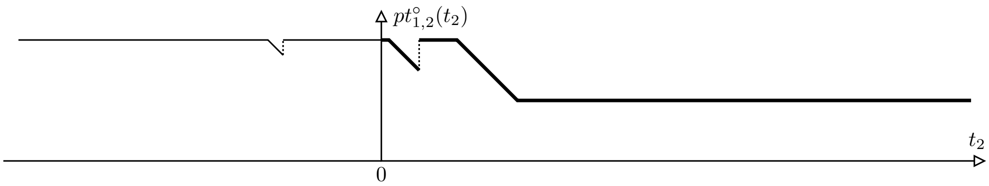


Figure 157: Optimal processing time $pt^{\circ}_{1,2}(t_2)$, under the assumption $\delta_1 = 1$ in state $[1\ 1\ 2\ t_2]^T$.

The conditioned cost-to-go $J^{\circ}_{1,1,2}(t_2 \mid \delta_1 = 1) = f(pt^{\circ}_{1,2}(t_2) + t_2) + g(pt^{\circ}_{1,2}(t_2))$, illustrated in figure 159, is provided by lemma 2. It is specified by the initial value 1.5, by the set { –7.5, –6.5, 0.5, 2.5, 5, 12.5, 14.5, 15, 16.1$\overline{6}$, 18.5, 19.5, 20.5 } of abscissae γ_i, $i = 1, \ldots, 12$, at which the slope changes, and by the set { 1, 0, 1, 0.5, 1, 1.5, 2, 3, 1.5, 2.5, 3, 4.5 } of slopes μ_i, $i = 1, \ldots, 12$, in the various intervals.

Case ii) in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{2,2}$, the following function

$$\alpha_{2,2}\ \max\{t_2 + st_{2,2} + \tau - dd_{2,2}\ , 0\} + \beta_2\,(pt_2^{\rm nom} - \tau) + sc_{2,2} + J^{\circ}_{1,2,2}(t_3)$$

that can be written as $f(pt_{2,2} + t_2) + g(pt_{2,2})$ being

$$f(pt_{2,2} + t_2) = \max\{pt_{2,2} + t_2 - 24\ , 0\} + J^{\circ}_{1,2,2}(pt_{2,2} + t_2)$$

$$g(pt_{2,2}) = \begin{cases} 1.5 \cdot (6 - pt_{2,2}) & pt_{2,2} \in [4, 6) \\ 0 & pt_{2,2} \notin [4, 6) \end{cases}$$

The function $pt^{\circ}_{2,2}(t_2) = \arg\min_{pt_{2,2}}\{f(pt_{2,2} + t_2) + g(pt_{2,2})\}$, with $4 \leq pt_{2,2} \leq 6$, is determined by applying lemma 1. It is (see figure 158)

$$pt^{\circ}_{2,2}(t_2) = x_{\rm e}(t_2) \qquad \text{with} \quad x_{\rm e}(t_2) = \begin{cases} 6 & t_2 < 13.5 \\ -t_2 + 19.5 & 13.5 \leq t_2 < 15.5 \\ 4 & t_2 \geq 15.5 \end{cases}$$

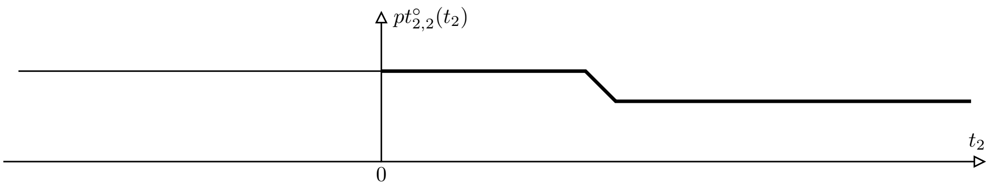


Figure 158: Optimal processing time $pt^{\circ}_{2,2}(t_2)$, under the assumption $\delta_2 = 1$ in state $[1\ 1\ 2\ t_2]^T$.

The conditioned cost-to-go $J^{\circ}_{1,1,2}(t_2 \mid \delta_2 = 1) = f(pt^{\circ}_{2,2}(t_2) + t_2) + g(pt^{\circ}_{2,2}(t_2))$, illustrated in figure 159, is provided by lemma 2. It is specified by the initial value 1, by the set { 0.5, 2, 3, 6.5, 13.5, 16.5, 18.5, 20, 26.25,

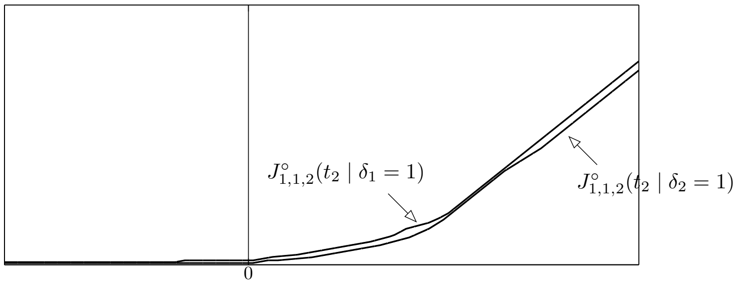


Figure 159: Conditioned costs-to-go $J^\circ_{1,1,2}(t_2 \mid \delta_1 = 1)$ and $J^\circ_{1,1,2}(t_2 \mid \delta_2 = 1)$ in state $[1\ 1\ 2\ t_2]^T$.

30 } of abscissae γ_i, $i = 1, \ldots, 10$, at which the slope changes, and by the set { 1, 0, 0.5, 1, 1.5, 2.5, 3.5, 4.5, 3.5, 4.5 } of slopes μ_i, $i = 1, \ldots, 10$, in the various intervals.

In order to find the optimal cost-to-go $J^\circ_{1,1,2}(t_2)$, it is necessary to carry out the following minimization

$$J^\circ_{1,1,2}(t_2) = \min \left\{ J^\circ_{1,1,2}(t_2 \mid \delta_1 = 1) \, , \, J^\circ_{1,1,2}(t_2 \mid \delta_2 = 1) \right\}$$

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 160.

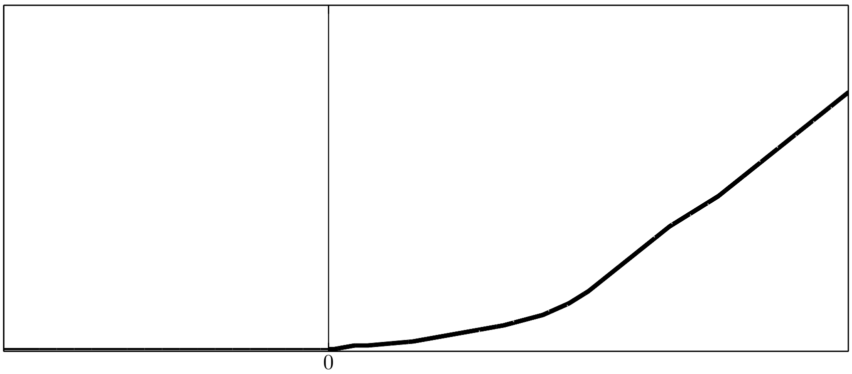


Figure 160: Optimal cost-to-go $J^\circ_{1,1,2}(t_2)$ in state $[1\ 1\ 2\ t_2]^T$.

The function $J^\circ_{1,1,2}(t_2)$ is specified by the initial value 1, by the set { 0.5, 2, 3, 6.5, 13.5, 16.5, 18.5, 20, 26.25, 30 } of abscissae γ_i, $i = 1, \ldots, 10$, at which the slope changes, and by the set { 1, 0, 0.5, 1, 1.5, 2.5, 3.5, 4.5, 3.5, 4.5 } of slopes μ_i, $i = 1, \ldots, 10$, in the various intervals.

Since $J^\circ_{1,1,2}(t_2 \mid \delta_2 = 1)$ is always the minimum (see again figure 159), the optimal control strategies for this state are

$$\delta^\circ_1(1, 1, 2, t_2) = 0 \quad \forall\, t_2 \qquad \delta^\circ_2(1, 1, 2, t_2) = 1 \quad \forall\, t_2$$

$$\tau^\circ(1, 1, 2, t_2) = \begin{cases} 6 & t_2 < 13.5 \\ -t_2 + 19.5 & 13.5 \leq t_2 < 15.5 \\ 4 & t_2 \geq 15.5 \end{cases}$$

The optimal control strategy $\tau^\circ(1, 1, 2, t_2)$ is illustrated in figure 161.

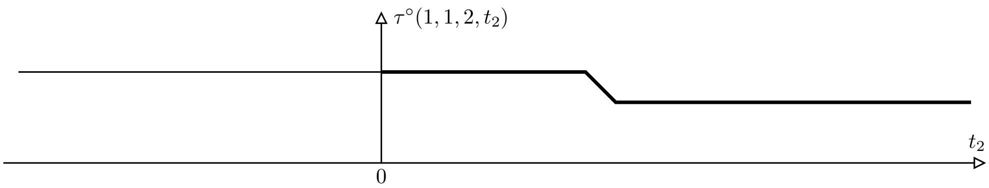


Figure 161: Optimal control strategy $\tau^\circ(1, 1, 2, t_2)$ in state $[1\ 1\ 2\ t_2]^T$.

Stage 2 – State $[1\ 1\ 1\ t_2]^T$ ($S4$)

In state $[1\ 1\ 1\ t_2]^T$, the cost function to be minimized, with respect to the (continuos) decision variable τ and to the (binary) decision variables δ_1 and δ_2 is

$$\delta_1\big[\alpha_{1,2}\ \max\{t_2+st_{1,1}+\tau-dd_{1,2}\ ,\ 0\}+\beta_1\,(pt_1^{\mathrm{nom}}-\tau)+sc_{1,1}+J^\circ_{2,1,1}(t_3)\big]+$$
$$+\ \delta_2\big[\alpha_{2,2}\ \max\{t_2+st_{1,2}+\tau-dd_{2,2}\ ,\ 0\}+\beta_2\,(pt_2^{\mathrm{nom}}-\tau)+sc_{1,2}+J^\circ_{1,2,2}(t_3)\big]$$

Case i) in which it is assumed $\delta_1=1$ (and $\delta_2=0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{1,2}$, the following function

$$\alpha_{1,2}\ \max\{t_2+st_{1,1}+\tau-dd_{1,2}\ ,\ 0\}+\beta_1\,(pt_1^{\mathrm{nom}}-\tau)+sc_{1,1}+J^\circ_{2,1,1}(t_3)$$

that can be written as $f(pt_{1,2}+t_2)+g(pt_{1,2})$ being

$$f(pt_{1,2}+t_2)=0.5\cdot\max\{pt_{1,2}+t_2-24\ ,\ 0\}+J^\circ_{2,1,1}(pt_{1,2}+t_2)$$

$$g(pt_{1,2})=\begin{cases}8-pt_{1,2} & pt_{1,2}\in[4,8)\\ 0 & pt_{1,2}\notin[4,8)\end{cases}$$

The function $pt^\circ_{1,2}(t_2)=\arg\min_{pt_{1,2}}\{f(pt_{1,2}+t_2)+g(pt_{1,2})\}$, with $4\le pt_{1,2}\le 8$, is determined by applying lemma 1. It is (see figure 162)

$$pt^\circ_{1,2}(t_2)=\begin{cases}x_{\mathrm{s}}(t_2) & t_2<-6\\ x_1(t_2) & -6\le t_2<3\\ x_{\mathrm{e}}(t_2) & t_2\ge 3\end{cases}\qquad\text{with}\quad x_{\mathrm{s}}(t_2)=\begin{cases}8 & t_2<-7\\ -t_2+1 & -7\le t_2<-6\end{cases},$$

$$x_1(t_2)=\begin{cases}8 & -6\le t_2<1\\ -t_2+9 & 1\le t_2<3\end{cases},\ \text{and}\quad x_{\mathrm{e}}(t_2)=\begin{cases}8 & 3\le t_2<5.5\\ -t_2+13.5 & 5.5\le t_2<9.5\\ 4 & t_2\ge 9.5\end{cases}$$

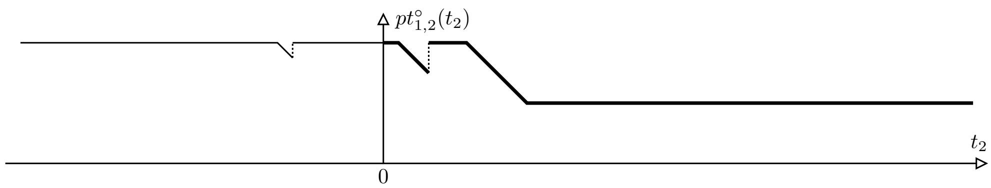


Figure 162: Optimal processing time $pt^\circ_{1,2}(t_2)$, under the assumption $\delta_1=1$ in state $[1\ 1\ 1\ t_2]^T$.

The conditioned cost-to-go $J^\circ_{1,1,1}(t_2\mid\delta_1=1)=f(pt^\circ_{1,2}(t_2)+t_2)+g(pt^\circ_{1,2}(t_2))$, illustrated in figure 164, is provided by lemma 2. It is specified by the initial value 0.5, by the set { –7, –6, 1, 3, 5.5, 13, 15, 15.5, 16.$\overline{6}$, 19, 20, 21 } of abscissae γ_i, $i=1,\ldots,12$, at which the slope changes, and by the set { 1, 0, 1, 0.5, 1, 1.5, 2, 3, 1.5, 2.5, 3, 4.5 } of slopes μ_i, $i=1,\ldots,12$, in the various intervals.

Case ii) in which it is assumed $\delta_2=1$ (and $\delta_1=0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{2,2}$, the following function

$$\alpha_{2,2}\ \max\{t_2+st_{1,2}+\tau-dd_{2,2}\ ,\ 0\}+\beta_2\,(pt_2^{\mathrm{nom}}-\tau)+sc_{1,2}+J^\circ_{1,2,2}(t_3)$$

that can be written as $f(pt_{2,2}+t_2)+g(pt_{2,2})$ being

$$f(pt_{2,2}+t_2)=\max\{pt_{2,2}+t_2-23\ ,\ 0\}+0.5+J^\circ_{1,2,2}(pt_{2,2}+t_2+1)$$

$$g(pt_{2,2})=\begin{cases}1.5\cdot(6-pt_{2,2}) & pt_{2,2}\in[4,6)\\ 0 & pt_{2,2}\notin[4,6)\end{cases}$$

The function $pt^{\circ}_{2,2}(t_2) = \arg\min_{pt_{2,2}}\{f(pt_{2,2}+t_2)+g(pt_{2,2})\}$, with $4 \le pt_{2,2} \le 6$, is determined by applying lemma 1. It is (see figure 163)

$$pt^{\circ}_{2,2}(t_2) = x_{\mathrm{e}}(t_2) \qquad \text{with} \quad x_{\mathrm{e}}(t_2) = \begin{cases} 6 & t_2 < 12.5 \\ -t_2 + 18.5 & 12.5 \le t_2 < 14.5 \\ 4 & t_2 \ge 14.5 \end{cases}$$

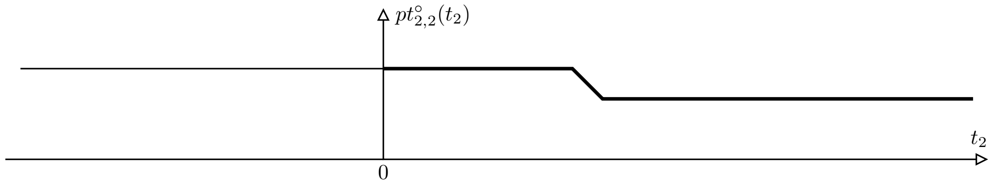


Figure 163: Optimal processing time $pt^{\circ}_{2,2}(t_2)$, under the assumption $\delta_2 = 1$ in state $[1\ 1\ 1\ t_2]^T$.

The conditioned cost-to-go $J^{\circ}_{1,1,1}(t_2 \mid \delta_2 = 1) = f(pt^{\circ}_{2,2}(t_2)+t_2)+g(pt^{\circ}_{2,2}(t_2))$, illustrated in figure 164, is provided by lemma 2. It is specified by the initial value 1.5, by the set { –0.5, 1, 2, 5.5, 12.5, 15.5, 17.5, 19, 25.25, 29 } of abscissae γ_i, $i = 1,\ldots,10$, at which the slope changes, and by the set { 1, 0, 0.5, 1, 1.5, 2.5, 3.5, 4.5, 3.5, 4.5 } of slopes μ_i, $i = 1,\ldots,10$, in the various intervals.

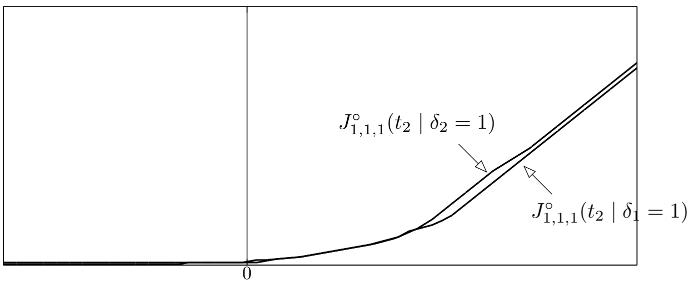


Figure 164: Conditioned costs-to-go $J^{\circ}_{1,1,1}(t_2 \mid \delta_1 = 1)$ and $J^{\circ}_{1,1,1}(t_2 \mid \delta_2 = 1)$ in state $[1\ 1\ 1\ t_2]^T$.

In order to find the optimal cost-to-go $J^{\circ}_{1,1,1}(t_2)$, it is necessary to carry out the following minimization

$$J^{\circ}_{1,1,1}(t_2) = \min\left\{J^{\circ}_{1,1,1}(t_2 \mid \delta_1 = 1)\,,\, J^{\circ}_{1,1,1}(t_2 \mid \delta_2 = 1)\right\}$$

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 165.

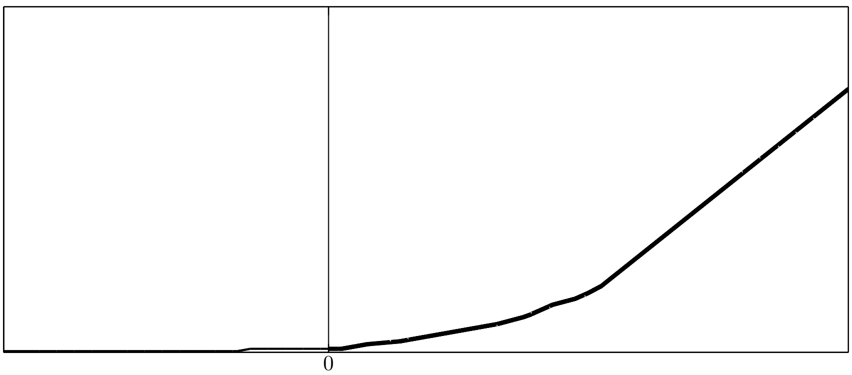


Figure 165: Optimal cost-to-go $J^{\circ}_{1,1,1}(t_2)$ in state $[1\ 1\ 1\ t_2]^T$.

The function $J^{\circ}_{1,1,1}(t_2)$ is specified by the initial value 0.5, by the set { –7, –6, 1, 3, 5.5, 13, 15, 15.5, 17.25, 19, 20, 21 } of abscissae γ_i, $i = 1,\ldots,10$, at which the slope changes, and by the set { 1, 0, 1, 0.5, 1, 1.5, 2, 2.5, 1.5, 2.5, 3, 4.5 } of slopes μ_i, $i = 1,\ldots,10$, in the various intervals.

Since $J^\circ_{1,1,1}(t_2 \mid \delta_1 = 1)$ is the minimum in $(-\infty, -6)$, in $[1, 15.5)$, and in $[17.25, +\infty)$, and $J^\circ_{1,1,1}(t_2 \mid \delta_2 = 1)$ is the minimum in $[-6, -1)$ and in $[15.5, 17.25)$, the optimal control strategies for this state are

$$\delta^\circ_1(1,1,1,t_2) = \begin{cases} 1 & t_2 < -6 \\ 0 & -6 \le t_2 < 1 \\ 1 & 1 \le t_2 < 15.5 \\ 0 & 15.5 \le t_2 < 17.25 \\ 1 & t_2 \ge 17.25 \end{cases} \qquad \delta^\circ_2(1,1,1,t_2) = \begin{cases} 0 & t_2 < -6 \\ 1 & -6 \le t_2 < 1 \\ 0 & 1 \le t_2 < 15.5 \\ 1 & 15.5 \le t_2 < 17.25 \\ 0 & t_2 \ge 17.25 \end{cases}$$

$$\tau^\circ(1,1,1,t_2) = \begin{cases} 8 & t_2 < -7 \\ -t_2 + 1 & -7 \le t_2 < -6 \\ 6 & -6 \le t_2 < 1 \\ -t_2 + 9 & 1 \le t_2 < 3 \\ 8 & 3 \le t_2 < 5.5 \\ -t_2 + 13.5 & 5.5 \le t_2 < 9.5 \\ 4 & t_2 \ge 9.5 \end{cases}$$

The optimal control strategy $\tau^\circ(1,1,1,t_2)$ is illustrated in figure 166.

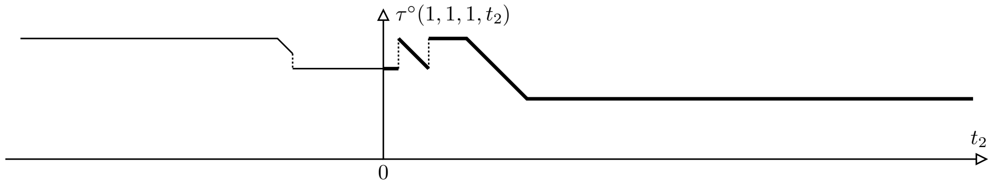


Figure 166: Optimal control strategy $\tau^\circ(1,1,1,t_2)$ in state $[1\ 1\ 1\ t_2]^T$.

Stage 2 – State $[2\ 0\ 1\ t_2]^T$ ($S3$)

In state $[2\ 0\ 1\ t_2]^T$, the cost function to be minimized, with respect to the (continuos) decision variable τ and to the (binary) decision variables δ_1 and δ_2 is

$$\delta_1\big[\alpha_{1,3}\ \max\{t_2 + st_{1,1} + \tau - dd_{1,3}\,,\,0\} + \beta_1\,(pt_1^{\text{nom}} - \tau) + sc_{1,1} + J^\circ_{3,0,1}(t_3)\big] +$$
$$+\,\delta_2\big[\alpha_{2,1}\ \max\{t_2 + st_{1,2} + \tau - dd_{2,1}\,,\,0\} + \beta_2\,(pt_2^{\text{nom}} - \tau) + sc_{1,2} + J^\circ_{2,1,2}(t_3)\big]$$

Case i) in which it is assumed $\delta_1 = 1$ (and $\delta_2 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{1,3}$, the following function

$$\alpha_{1,3}\ \max\{t_2 + st_{1,1} + \tau - dd_{1,3}\,,\,0\} + \beta_1\,(pt_1^{\text{nom}} - \tau) + sc_{1,1} + J^\circ_{3,0,1}(t_3)$$

that can be written as $f(pt_{1,3} + t_2) + g(pt_{1,3})$ being

$$f(pt_{1,3} + t_2) = 1.5 \cdot \max\{pt_{1,3} + t_2 - 29\,,\,0\} + J^\circ_{3,0,1}(pt_{1,3} + t_2)$$

$$g(pt_{1,3}) = \begin{cases} 8 - pt_{1,3} & pt_{1,3} \in [4,8) \\ 0 & pt_{1,3} \notin [4,8) \end{cases}$$

The function $pt^\circ_{1,3}(t_2) = \arg\min_{pt_{1,3}}\{f(pt_{1,3} + t_2) + g(pt_{1,3})\}$, with $4 \le pt_{1,3} \le 8$, is determined by applying lemma 1. It is (see figure 167)

$$pt^\circ_{1,3}(t_2) = \begin{cases} x_{\text{s}}(t_2) & t_2 < -4 \\ x_{\text{e}}(t_2) & t_2 \ge -4 \end{cases} \qquad \text{with} \quad x_{\text{s}}(t_2) = \begin{cases} 8 & t_2 < -5 \\ -t_2 + 3 & -5 \le t_2 < -4 \end{cases} ,$$

$$\text{and} \quad x_{\text{e}}(t_2) = \begin{cases} 8 & -4 \le t_2 < 3 \\ -t_2 + 11 & 3 \le t_2 < 7 \\ 4 & t_2 \ge 7 \end{cases}$$

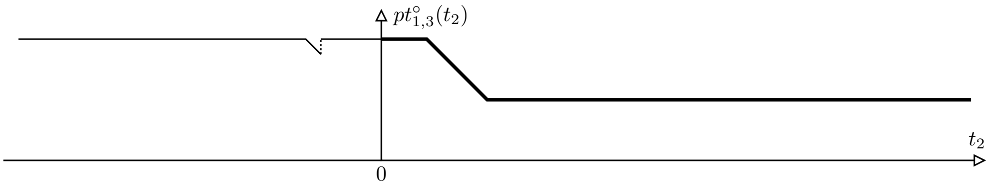


Figure 167: Optimal processing time $pt^\circ_{1,3}(t_2)$, under the assumption $\delta_1 = 1$ in state $[2\ 0\ 1\ t_2]^T$.

The conditioned cost-to-go $J^\circ_{2,0,1}(t_2 \mid \delta_1 = 1) = f(pt^\circ_{1,3}(t_2) + t_2) + g(pt^\circ_{1,3}(t_2))$, illustrated in figure 169, is provided by lemma 2. It is specified by the initial value 0.5, by the set { –5, –4, 3, 9.5, 12, 19, 25 } of abscissae γ_i, $i = 1, \dots, 7$, at which the slope changes, and by the set { 1, 0, 1, 1.5, 3.5, 4.5, 6 } of slopes μ_i, $i = 1, \dots, 7$, in the various intervals.

Case ii) in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{2,1}$, the following function

$$\alpha_{2,1}\ \max\{t_2 + st_{1,2} + \tau - dd_{2,1}\,,\,0\} + \beta_2\,(pt_2^{\mathrm{nom}} - \tau) + sc_{1,2} + J^\circ_{2,1,2}(t_3)$$

that can be written as $f(pt_{2,1} + t_2) + g(pt_{2,1})$ being

$$f(pt_{2,1} + t_2) = 2 \cdot \max\{pt_{2,1} + t_2 - 20\,,\,0\} + 0.5 + J^\circ_{2,1,2}(pt_{2,1} + t_2 + 1)$$

$$g(pt_{2,1}) = \begin{cases} 1.5 \cdot (6 - pt_{2,1}) & pt_{2,1} \in [4,6) \\ 0 & pt_{2,1} \notin [4,6) \end{cases}$$

The function $pt^\circ_{2,1}(t_2) = \arg\min_{pt_{2,1}} \{f(pt_{2,1} + t_2) + g(pt_{2,1})\}$, with $4 \le pt_{2,1} \le 6$, is determined by applying lemma 1. It is (see figure 168)

$$pt^\circ_{2,1}(t_2) = x_\mathrm{e}(t_2) \qquad \text{with} \quad x_\mathrm{e}(t_2) = \begin{cases} 6 & t_2 < 11 \\ -t_2 + 17 & 11 \le t_2 < 13 \\ 4 & t_2 \ge 13 \end{cases}$$

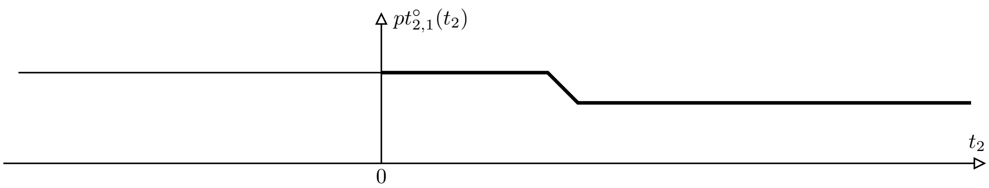


Figure 168: Optimal processing time $pt^\circ_{2,1}(t_2)$, under the assumption $\delta_2 = 1$ in state $[2\ 0\ 1\ t_2]^T$.

The conditioned cost-to-go $J^\circ_{2,0,1}(t_2 \mid \delta_2 = 1) = f(pt^\circ_{2,1}(t_2) + t_2) + g(pt^\circ_{2,1}(t_2))$, illustrated in figure 169, is provided by lemma 2. It is specified by the initial value 1.5, by the set { 1.5, 3, 4, 7.5, 11, 15, 15.5, 16, 17.5, 19.1$\overline{6}$, 19.5, 23.75, 25 } of abscissae γ_i, $i = 1, \dots, 13$, at which the slope changes, and by the set { 1, 0, 0.5, 1, 1.5, 2, 3, 5, 6, 4.5, 6, 5, 6 } of slopes μ_i, $i = 1, \dots, 13$, in the various intervals.

In order to find the optimal cost-to-go $J^\circ_{2,0,1}(t_2)$, it is necessary to carry out the following minimization

$$J^\circ_{2,0,1}(t_2) = \min\left\{J^\circ_{2,0,1}(t_2 \mid \delta_1 = 1)\,,\, J^\circ_{2,0,1}(t_2 \mid \delta_2 = 1)\right\}$$

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 170.

The function $J^\circ_{2,0,1}(t_2)$ is specified by the initial value 0.5, by the set { –5, –4, 3, 5, 7.5, 11, 15, 15.5, 16, 17.5, 19.1$\overline{6}$, 19.5, 21.$\overline{3}$, 25 } of abscissae γ_i, $i = 1, \dots, 14$, at which the slope changes, and by the set { 1, 0, 1, 0.5, 1, 1.5, 2, 3, 5, 6, 4.5, 6, 4.5, 6 } of slopes μ_i, $i = 1, \dots, 14$, in the various intervals.

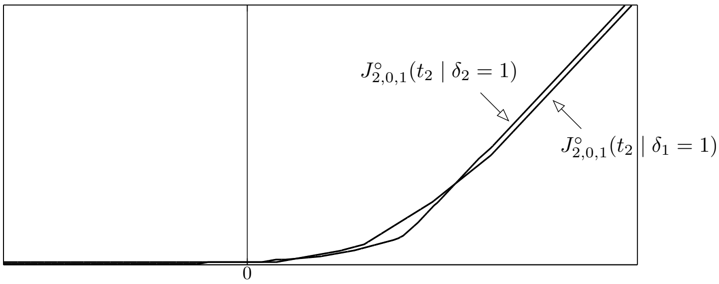


Figure 169: Conditioned costs-to-go $J^\circ_{2,0,1}(t_2 \mid \delta_1 = 1)$ and $J^\circ_{2,0,1}(t_2 \mid \delta_2 = 1)$ in state $[2\ 0\ 1\ t_2]^T$.

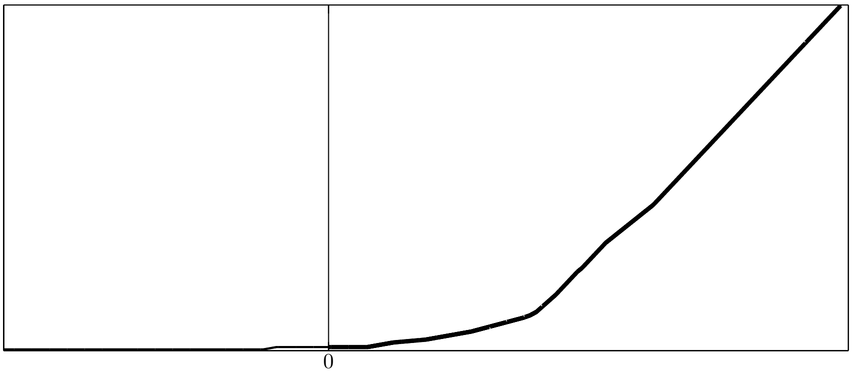


Figure 170: Optimal cost-to-go $J^\circ_{2,0,1}(t_2)$ in state $[2\ 0\ 1\ t_2]^T$.

Since $J^\circ_{2,0,1}(t_2 \mid \delta_1 = 1)$ is the minimum in $(-\infty, -4)$, in $[3, 5)$, and in $[21.\overline{3}, +\infty)$, and $J^\circ_{2,0,1}(t_2 \mid \delta_2 = 1)$ is the minimum in $[-4, 3)$ and in $[5, 21.\overline{3})$, the optimal control strategies for this state are

$$\delta_1^\circ(2,0,1,t_2) = \begin{cases} 1 & t_2 < -4 \\ 0 & -4 \le t_2 < 3 \\ 1 & 3 \le t_2 < 5 \\ 0 & 5 \le t_2 < 21.\overline{3} \\ 1 & t_2 \ge 21.\overline{3} \end{cases} \qquad \delta_2^\circ(2,0,1,t_2) = \begin{cases} 0 & t_2 < -4 \\ 1 & -4 \le t_2 < 3 \\ 0 & 3 \le t_2 < 5 \\ 1 & 5 \le t_2 < 21.\overline{3} \\ 0 & t_2 \ge 21.\overline{3} \end{cases}$$

$$\tau^\circ(2,0,1,t_2) = \begin{cases} 8 & t_2 < -5 \\ -t_2 + 3 & -5 \le t_2 < -4 \\ 6 & -4 \le t_2 < 3 \\ -t_2 + 11 & 3 \le t_2 < 5 \\ 6 & 5 \le t_2 < 11 \\ -t_2 + 17 & 11 \le t_2 < 13 \\ 4 & t_2 \ge 13 \end{cases}$$

The optimal control strategy $\tau^\circ(2,0,1,t_2)$ is illustrated in figure 171.

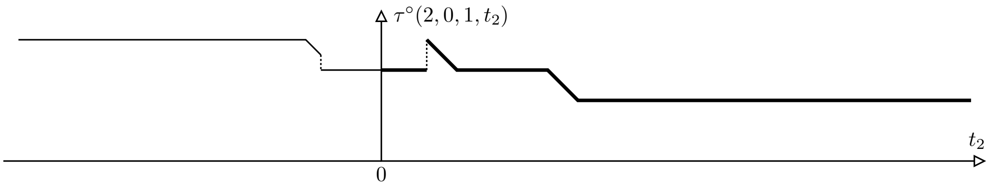


Figure 171: Optimal control strategy $\tau^\circ(2,0,1,t_2)$ in state $[2\ 0\ 1\ t_2]^T$.

Stage 1 – State $[0\ 1\ 2\ t_1]^T$ ($S2$)

In state $[0\ 1\ 2\ t_1]^T$, the cost function to be minimized, with respect to the (continuos) decision variable τ and to the (binary) decision variables δ_1 and δ_2 is

$$\delta_1\big[\alpha_{1,1}\ \max\{t_1+st_{2,1}+\tau-dd_{1,1}\,,\,0\}+\beta_1\,(pt_1^{\text{nom}}-\tau)+sc_{2,1}+J^\circ_{1,1,1}(t_2)\big]+$$
$$+\,\delta_2\big[\alpha_{2,2}\ \max\{t_1+st_{2,2}+\tau-dd_{2,2}\,,\,0\}+\beta_2\,(pt_2^{\text{nom}}-\tau)+sc_{2,2}+J^\circ_{0,2,2}(t_2)\big]$$

Case i) in which it is assumed $\delta_1=1$ (and $\delta_2=0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{1,1}$, the following function

$$\alpha_{1,1}\ \max\{t_1+st_{2,1}+\tau-dd_{1,1}\,,\,0\}+\beta_1\,(pt_1^{\text{nom}}-\tau)+sc_{2,1}+J^\circ_{1,1,1}(t_2)$$

that can be written as $f(pt_{1,1}+t_1)+g(pt_{1,1})$ being

$$f(pt_{1,1}+t_1)=0.75\cdot\max\{pt_{1,1}+t_1-18.5\,,\,0\}+1+J^\circ_{1,1,1}(pt_{1,1}+t_1+0.5)$$

$$g(pt_{1,1})=\begin{cases}8-pt_{1,1} & pt_{1,1}\in[4,8)\\ 0 & pt_{1,1}\notin[4,8)\end{cases}$$

The function $pt^\circ_{1,1}(t_1)=\arg\min_{pt_{1,1}}\{f(pt_{1,1}+t_1)+g(pt_{1,1})\}$, with $4\le pt_{1,1}\le 8$, is determined by applying lemma 1. It is (see figure 172)

$$pt^\circ_{1,1}(t_1)=\begin{cases}x_{\text{s}}(t_1) & t_1<-14.5\\ x_1(t_1) & -14.5\le t_1<-5.5\\ x_{\text{e}}(t_1) & t_1\ge-5.5\end{cases}\qquad\text{with}\quad x_{\text{s}}(t_1)=\begin{cases}8 & t_1<-15.5\\ -t_1-7.5 & -15.5\le t_1<-14.5\end{cases},$$

$$x_1(t_1)=\begin{cases}8 & -14.5\le t_1<-7.5\\ -t_1+0.5 & -7.5\le t_1<-5.5\end{cases}\quad,\ \text{and}\quad x_{\text{e}}(t_1)=\begin{cases}8 & -5.5\le t_1<-3\\ -t_1+5 & -3\le t_1<1\\ 4 & t_1\ge 1\end{cases}$$

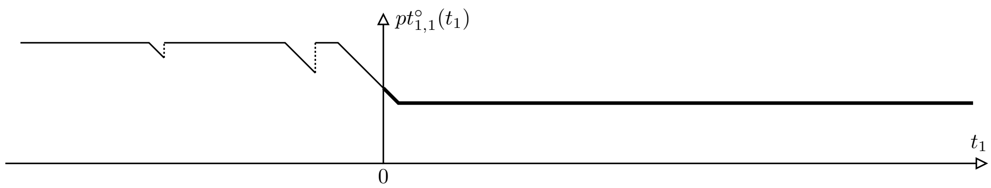


Figure 172: Optimal processing time $pt^\circ_{1,1}(t_1)$, under the assumption $\delta_1=1$ in state $[0\ 1\ 2\ t_1]^T$.

The conditioned cost-to-go $J^\circ_{0,1,2}(t_1\mid\delta_1=1)=f(pt^\circ_{1,1}(t_1)+t_1)+g(pt^\circ_{1,1}(t_1))$, illustrated in figure 174, is provided by lemma 2. It is specified by the initial value 1.5, by the set { –15.5, –14.5, –7.5, –5.5, –3, 8.5, 10.5, 11, 12.75, 14.5, 15.5, 16.5 } of abscissae γ_i, $i=1,\ldots,12$, at which the slope changes, and by the set { 1, 0, 1, 0.5, 1, 1.5, 2, 2.5, 1.5, 3.25, 3.75, 5.25 } of slopes μ_i, $i=1,\ldots,12$, in the various intervals.

Case ii) in which it is assumed $\delta_2=1$ (and $\delta_1=0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{2,2}$, the following function

$$\alpha_{2,2}\ \max\{t_1+st_{2,2}+\tau-dd_{2,2}\,,\,0\}+\beta_2\,(pt_2^{\text{nom}}-\tau)+sc_{2,2}+J^\circ_{0,2,2}(t_2)$$

that can be written as $f(pt_{2,2}+t_1)+g(pt_{2,2})$ being

$$f(pt_{2,2}+t_1)=\max\{pt_{2,2}+t_1-24\,,\,0\}+J^\circ_{0,2,2}(pt_{2,2}+t_1)$$

$$g(pt_{2,2})=\begin{cases}1.5\cdot(6-pt_{2,2}) & pt_{2,2}\in[4,6)\\ 0 & pt_{2,2}\notin[4,6)\end{cases}$$

The function $pt^{\circ}_{2,2}(t_1) = \arg\min_{pt_{2,2}} \{f(pt_{2,2} + t_1) + g(pt_{2,2})\}$, with $4 \leq pt_{2,2} \leq 6$, is determined by applying lemma 1. It is (see figure 173)

$$pt^{\circ}_{2,2}(t_1) = x_{\mathrm{e}}(t_1) \qquad \text{with} \quad x_{\mathrm{e}}(t_1) = \begin{cases} 6 & t_1 < 8.5 \\ -t_1 + 14.5 & 8.5 \leq t_1 < 10.5 \\ 4 & t_1 \geq 10.5 \end{cases}$$

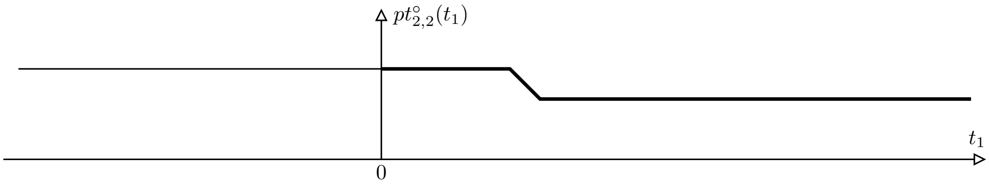


Figure 173: Optimal processing time $pt^{\circ}_{2,2}(t_1)$, under the assumption $\delta_2 = 1$ in state $[0\ 1\ 2\ t_1]^T$.

The conditioned cost-to-go $J^{\circ}_{0,1,2}(t_1 \mid \delta_2 = 1) = f(pt^{\circ}_{2,2}(t_1) + t_1) + g(pt^{\circ}_{2,2}(t_1))$, illustrated in figure 174, is provided by lemma 2. It is specified by the initial value 1, by the set $\{$ –7.5, –6, –5, –1.5, 8.5, 10.5, 11.5, 12.5, 14.5, 20, 25.25, 30 $\}$ of abscissae γ_i, $i = 1, \ldots, 12$, at which the slope changes, and by the set $\{$ 1, 0, 0.5, 1, 1.5, 1.75, 2.25, 3.25, 4.25, 5.25, 4.25, 5.25 $\}$ of slopes μ_i, $i = 1, \ldots, 12$, in the various intervals.

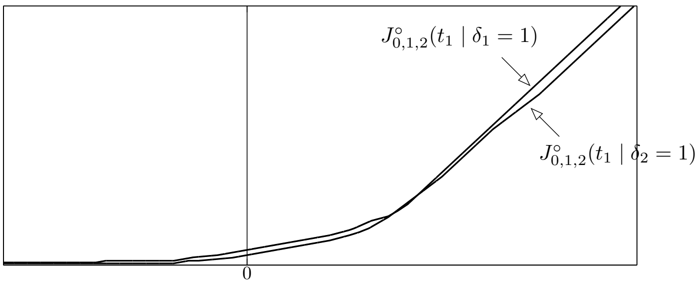


Figure 174: Conditioned costs-to-go $J^{\circ}_{0,1,2}(t_1 \mid \delta_1 = 1)$ and $J^{\circ}_{0,1,2}(t_1 \mid \delta_2 = 1)$ in state $[0\ 1\ 2\ t_1]^T$.

In order to find the optimal cost-to-go $J^{\circ}_{0,1,2}(t_1)$, it is necessary to carry out the following minimization

$$J^{\circ}_{0,1,2}(t_1) = \min \left\{ J^{\circ}_{0,1,2}(t_1 \mid \delta_1 = 1) \, , \, J^{\circ}_{0,1,2}(t_1 \mid \delta_2 = 1) \right\}$$

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 175.

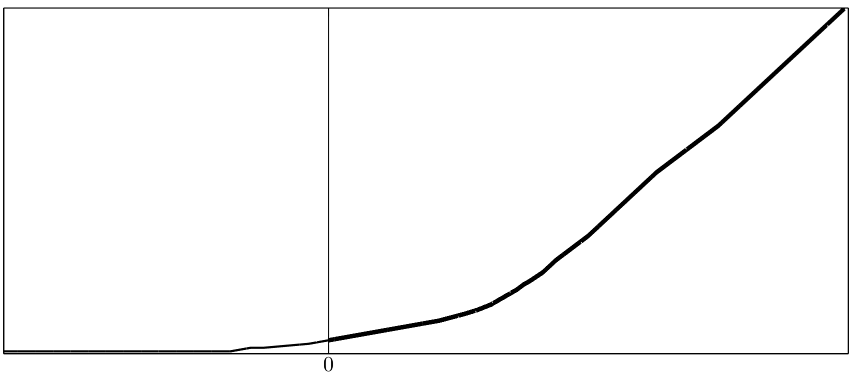


Figure 175: Optimal cost-to-go $J^{\circ}_{0,1,2}(t_1)$ in state $[0\ 1\ 2\ t_1]^T$.

The function $J^{\circ}_{0,1,2}(t_1)$ is specified by the initial value 1, by the set $\{$ –7.5, –6, –5, –1.5, 8.5, 10.5, 11.5, 12.5, 14.5, 15, 15.5, 16.5, 17.5, 20, 25.25, 30 $\}$ of abscissae γ_i, $i = 1, \ldots, 16$, at which the slope changes, and by the set $\{$ 1, 0, 0.5, 1, 1.5, 1.75, 2.25, 3.25, 4.25, 3.25, 3.75, 5.25, 4.25, 5.25, 4.25, 5.25 $\}$ of slopes μ_i, $i = 1, \ldots, 16$, in the various intervals.

Since $J^{\circ}_{0,1,2}(t_1 \mid \delta_1 = 1)$ is the minimum in $[15, 17.5)$, and $J^{\circ}_{0,1,2}(t_1 \mid \delta_2 = 1)$ is the minimum in $(-\infty, 15)$ and in $[17.5, +\infty)$, the optimal control strategies for this state are

$$\delta_1^{\circ}(0,1,2,t_1) = \begin{cases} 0 & t_1 < 15 \\ 1 & 15 \le t_1 < 17.5 \\ 0 & t_1 \ge 17.5 \end{cases} \qquad \delta_2^{\circ}(0,1,2,t_1) = \begin{cases} 1 & t_1 < 15 \\ 0 & 15 \le t_1 < 17.5 \\ 1 & t_1 \ge 17.5 \end{cases}$$

$$\tau^{\circ}(0,1,2,t_1) = \begin{cases} 6 & t_1 < 8.5 \\ -t_1 + 14.5 & 8.5 \le t_1 < 10.5 \\ 4 & t_1 \ge 10.5 \end{cases}$$

The optimal control strategy $\tau^{\circ}(0,1,2,t_1)$ is illustrated in figure 176.

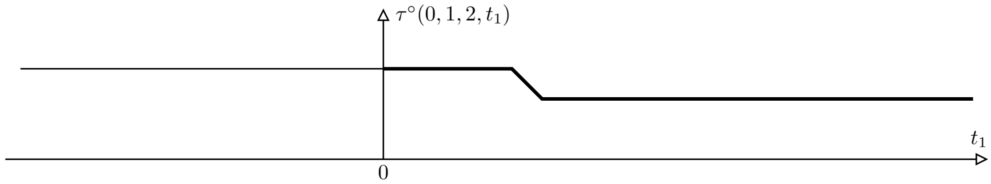


Figure 176: Optimal control strategy $\tau^{\circ}(0,1,2,t_1)$ in state $[0\ 1\ 2\ t_1]^T$.

Stage 1 – State $[\mathbf{1\ 0\ 1}\ \boldsymbol{t_1}]^{\boldsymbol{T}}$ ($S1$)

In state $[1\ 0\ 1\ t_1]^T$, the cost function to be minimized, with respect to the (continuos) decision variable τ and to the (binary) decision variables δ_1 and δ_2 is

$$\begin{aligned} &\delta_1 \big[\alpha_{1,2}\ \max\{t_1 + st_{1,1} + \tau - dd_{1,2}\,,\, 0\} + \beta_1\,(pt_1^{\text{nom}} - \tau) + sc_{1,1} + J^{\circ}_{2,0,1}(t_2)\big] + \\ &+ \delta_2 \big[\alpha_{2,1}\ \max\{t_1 + st_{1,2} + \tau - dd_{2,1}\,,\, 0\} + \beta_2\,(pt_2^{\text{nom}} - \tau) + sc_{1,2} + J^{\circ}_{1,1,2}(t_2)\big] \end{aligned}$$

Case i) in which it is assumed $\delta_1 = 1$ (and $\delta_2 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{1,2}$, the following function

$$\alpha_{1,2}\ \max\{t_1 + st_{1,1} + \tau - dd_{1,2}\,,\, 0\} + \beta_1\,(pt_1^{\text{nom}} - \tau) + sc_{1,1} + J^{\circ}_{2,0,1}(t_2)$$

that can be written as $f(pt_{1,2} + t_1) + g(pt_{1,2})$ being

$$f(pt_{1,2} + t_1) = 0.5 \cdot \max\{pt_{1,2} + t_1 - 24\,,\, 0\} + J^{\circ}_{2,0,1}(pt_{1,2} + t_1)$$

$$g(pt_{1,2}) = \begin{cases} 8 - pt_{1,2} & pt_{1,2} \in [4, 8) \\ 0 & pt_{1,2} \notin [4, 8) \end{cases}$$

The function $pt^{\circ}_{1,2}(t_1) = \arg\min_{pt_{1,2}} \{f(pt_{1,2} + t_1) + g(pt_{1,2})\}$, with $4 \le pt_{1,2} \le 8$, is determined by applying lemma 1. It is (see figure 177)

$$pt^{\circ}_{1,2}(t_1) = \begin{cases} x_{\text{s}}(t_1) & t_1 < -12 \\ x_1(t_1) & -12 \le t_1 < -3 \\ x_{\text{e}}(t_1) & t_1 \ge -3 \end{cases} \qquad \text{with} \quad x_{\text{s}}(t_1) = \begin{cases} 8 & t_1 < -13 \\ -t_1 - 5 & -13 \le t_1 < -12 \end{cases},$$

$$x_1(t_1) = \begin{cases} 8 & -12 \le t_1 < -5 \\ -t_1 + 3 & -5 \le t_1 < -3 \end{cases}, \text{ and } \quad x_{\text{e}}(t_1) = \begin{cases} 8 & -3 \le t_1 < -0.5 \\ -t_1 + 7.5 & -0.5 \le t_1 < 3.5 \\ 4 & t_1 \ge 3.5 \end{cases}$$

The conditioned cost-to-go $J^{\circ}_{1,0,1}(t_1 \mid \delta_1 = 1) = f(pt^{\circ}_{1,2}(t_1) + t_1) + g(pt^{\circ}_{1,2}(t_1))$, illustrated in figure 179, is provided by lemma 2. It is specified by the initial value 0.5, by the set { –13, –12, –5, –3, –0.5, 7, 11, 11.5, 12, 13.5, $15.1\overline{6}$, 15.5, $17.\overline{3}$, 20, 21 } of abscissae γ_i, $i = 1, \ldots, 15$, at which the slope changes, and by the set { 1, 0, 1, 0.5, 1, 1.5, 2, 3, 5, 6, 4.5, 6, 4.5, 5, 6.5 } of slopes μ_i, $i = 1, \ldots, 15$, in the various intervals.

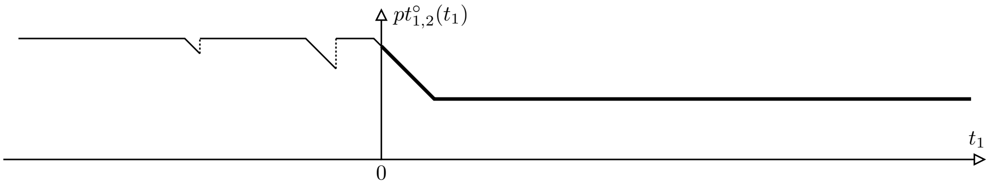


Figure 177: Optimal processing time $pt^{\circ}_{1,2}(t_1)$, under the assumption $\delta_1 = 1$ in state $[1\ 0\ 1\ t_1]^T$.

Case ii) in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{2,1}$, the following function

$$\alpha_{2,1}\ \max\{t_1 + st_{1,2} + \tau - dd_{2,1}\ ,\ 0\} + \beta_2\,(pt_2^{\text{nom}} - \tau) + sc_{1,2} + J^{\circ}_{1,1,2}(t_2)$$

that can be written as $f(pt_{2,1} + t_1) + g(pt_{2,1})$ being

$$f(pt_{2,1} + t_1) = 2 \cdot \max\{pt_{2,1} + t_1 - 20\ ,\ 0\} + 0.5 + J^{\circ}_{1,1,2}(pt_{2,1} + t_1 + 1)$$

$$g(pt_{2,1}) = \begin{cases} 1.5 \cdot (6 - pt_{2,1}) & pt_{2,1} \in [4, 6) \\ 0 & pt_{2,1} \notin [4, 6) \end{cases}$$

The function $pt^{\circ}_{2,1}(t_1) = \arg\min_{pt_{2,1}}\{f(pt_{2,1} + t_1) + g(pt_{2,1})\}$, with $4 \le pt_{2,1} \le 6$, is determined by applying lemma 1. It is (see figure 178)

$$pt^{\circ}_{2,1}(t_1) = x_{\text{e}}(t_1) \qquad \text{with} \quad x_{\text{e}}(t_1) = \begin{cases} 6 & t_1 < 6.5 \\ -t_1 + 12.5 & 6.5 \le t_1 < 8.5 \\ 4 & t_1 \ge 8.5 \end{cases}$$

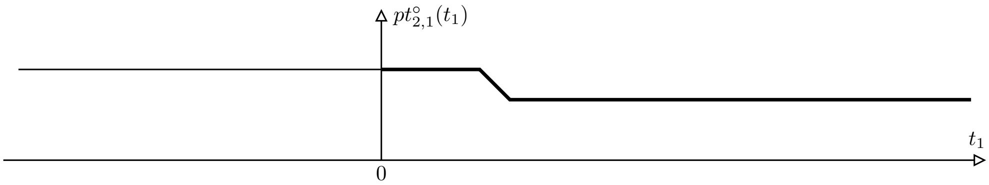


Figure 178: Optimal processing time $pt^{\circ}_{2,1}(t_1)$, under the assumption $\delta_2 = 1$ in state $[1\ 0\ 1\ t_1]^T$.

The conditioned cost-to-go $J^{\circ}_{1,0,1}(t_1 \mid \delta_2 = 1) = f(pt^{\circ}_{2,1}(t_1) + t_1) + g(pt^{\circ}_{2,1}(t_1))$, illustrated in figure 179, is provided by lemma 2. It is specified by the initial value 1.5, by the set { –6.5, –5, –4, –0.5, 6.5, 11.5, 13.5, 15, 16, 21.25, 25 } of abscissae γ_i, $i = 1, \ldots, 11$, at which the slope changes, and by the set { 1, 0, 0.5, 1, 1.5, 2.5, 3.5, 4.5, 6.5, 5.5, 6.5 } of slopes μ_i, $i = 1, \ldots, 11$, in the various intervals.

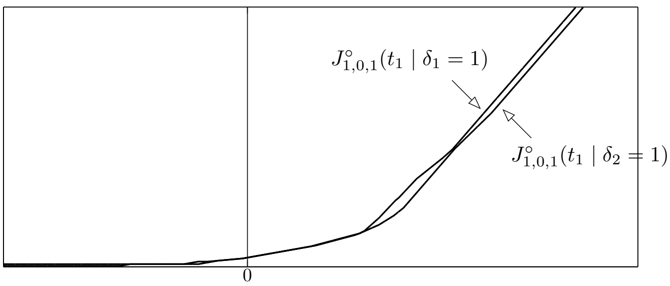


Figure 179: Conditioned costs-to-go $J^{\circ}_{1,0,1}(t_1 \mid \delta_1 = 1)$ and $J^{\circ}_{1,0,1}(t_1 \mid \delta_2 = 1)$ in state $[1\ 0\ 1\ t_1]^T$.

In order to find the optimal cost-to-go $J^{\circ}_{1,0,1}(t_1)$, it is necessary to carry out the following minimization

$$J^{\circ}_{1,0,1}(t_1) = \min\left\{J^{\circ}_{1,0,1}(t_1 \mid \delta_1 = 1)\ ,\ J^{\circ}_{1,0,1}(t_1 \mid \delta_2 = 1)\right\}$$

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 180.

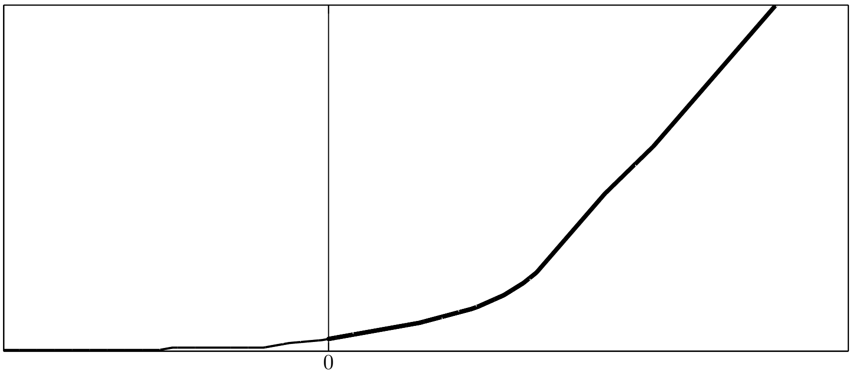


Figure 180: Optimal cost-to-go $J^{\circ}_{1,0,1}(t_1)$ in state $[1\ 0\ 1\ t_1]^T$.

The function $J^{\circ}_{1,0,1}(t_1)$ is specified by the initial value 0.5, by the set { –13, –12, –5, –3, –0.5, 7, 11, 11.5, 13.5, 15, 16, 21.25, 25 } of abscissae γ_i, $i = 1, \ldots, 13$, at which the slope changes, and by the set { 1, 0, 1, 0.5, 1, 1.5, 2, 2.5, 3.5, 4.5, 6.5, 5.5, 6.5 } of slopes μ_i, $i = 1, \ldots, 13$, in the various intervals.

Since $J^{\circ}_{1,0,1}(t_1 \mid \delta_1 = 1)$ is the minimum in $(-\infty, 12)$ and in $[-5, 11.5)$, and $J^{\circ}_{1,0,1}(t_1 \mid \delta_2 = 1)$ is the minimum in $[-12, -5)$ and in $[11.5, +\infty)$, the optimal control strategies for this state are

$$\delta^{\circ}_1(1,0,1,t_1) = \begin{cases} 1 & t_1 < -12 \\ 0 & -12 \le t_1 < -5 \\ 1 & -5 \le t_1 < 11.5 \\ 0 & t_1 \ge 11.5 \end{cases} \qquad \delta^{\circ}_2(1,0,1,t_1) = \begin{cases} 0 & t_1 < -12 \\ 1 & -12 \le t_1 < -5 \\ 0 & -5 \le t_1 < 11.5 \\ 1 & t_1 \ge 11.5 \end{cases}$$

$$\tau^{\circ}(1,0,1,t_1) = \begin{cases} 8 & t_1 < -13 \\ -t_1 - 5 & -13 \le t_1 < -12 \\ 6 & -12 \le t_1 < -5 \\ -t_1 + 3 & -5 \le t_1 < -3 \\ 8 & -3 \le t_1 < -0.5 \\ -t_1 + 7.5 & -0.5 \le t_1 < 3.5 \\ 4 & t_1 \ge 3.5 \end{cases}$$

The optimal control strategy $\tau^{\circ}(1,0,1,t_1)$ is illustrated in figure 181.

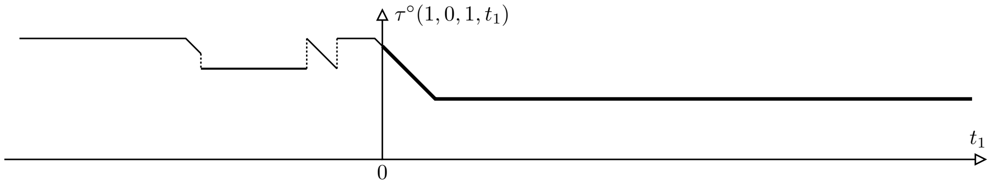


Figure 181: Optimal control strategy $\tau^{\circ}(1,0,1,t_1)$ in state $[1\ 0\ 1\ t_1]^T$.

Stage 0 – State $[\mathbf{0\ 0\ 0}\ \boldsymbol{t_0}]^{\boldsymbol{T}}$ ($S0$)

In the initial state $[0\ 0\ 0\ t_0]^T$, the cost function to be minimized, with respect to the (continuos) decision variable τ and to the (binary) decision variables δ_1 and δ_2 is

$$\begin{aligned} &\delta_1 \big[\alpha_{1,1}\ \max\{t_0 + st_{0,1} + \tau - dd_{1,1}\ ,\ 0\} + \beta_1\,(pt_1^{\text{nom}} - \tau) + sc_{0,1} + J^{\circ}_{1,0,1}(t_1)\big] + \\ &+ \delta_2 \big[\alpha_{2,1}\ \max\{t_0 + st_{0,2} + \tau - dd_{2,1}\ ,\ 0\} + \beta_2\,(pt_2^{\text{nom}} - \tau) + sc_{0,2} + J^{\circ}_{0,1,2}(t_1)\big] \end{aligned}$$

Case i) in which it is assumed $\delta_1 = 1$ (and $\delta_2 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{1,1}$, the following function

$$\alpha_{1,1}\ \max\{t_0 + st_{0,1} + \tau - dd_{1,1}\ ,\ 0\} + \beta_1\,(pt_1^{\text{nom}} - \tau) + sc_{0,1} + J^{\circ}_{1,0,1}(t_1)$$

that can be written as $f(pt_{1,1}+t_0)+g(pt_{1,1})$ being

$$f(pt_{1,1}+t_0) = 0.75 \cdot \max\{pt_{1,1}+t_0-19\,,\,0\} + J^{\circ}_{1,0,1}(pt_{1,1}+t_0)$$

$$g(pt_{1,1}) = \begin{cases} 8-pt_{1,1} & pt_{1,1} \in [4,8) \\ 0 & pt_{1,1} \notin [4,8) \end{cases}$$

The function $pt^{\circ}_{1,1}(t_0) = \arg\min_{pt_{1,1}}\{f(pt_{1,1}+t_0)+g(pt_{1,1})\}$, with $4 \le pt_{1,1} \le 8$, is determined by applying lemma 1. It is (see figure 182)

$$pt^{\circ}_{1,1}(t_0) = \begin{cases} x_{\mathrm{s}}(t_0) & t_0 < -20 \\ x_1(t_0) & -20 \le t_0 < -11 \\ x_{\mathrm{e}}(t_0) & t_0 \ge -11 \end{cases} \qquad \text{with} \quad x_{\mathrm{s}}(t_0) = \begin{cases} 8 & t_0 < -21 \\ -t_0-13 & -21 \le t_0 < -20 \end{cases} \;,$$

$$x_1(t_0) = \begin{cases} 8 & -20 \le t_0 < -13 \\ -t_0-5 & -13 \le t_0 < -11 \end{cases} \;, \text{ and } \quad x_{\mathrm{e}}(t_0) = \begin{cases} 8 & -11 \le t_0 < -8.5 \\ -t_0-0.5 & -8.5 \le t_0 < -4.5 \\ 4 & t_0 \ge -4.5 \end{cases}$$

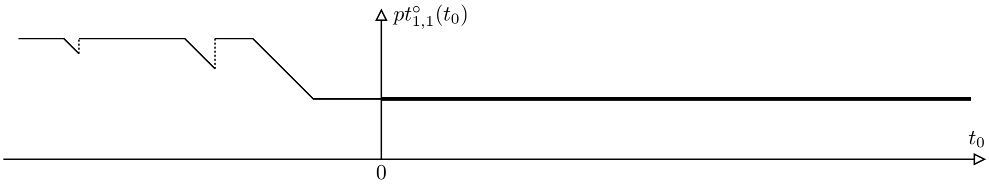


Figure 182: Optimal processing time $pt^{\circ}_{1,1}(t_0)$, under the assumption $\delta_1 = 1$ in the initial state $[0\ 0\ 0\ t_0]^T$.

The conditioned cost-to-go $J^{\circ}_{0,0,0}(t_0 \mid \delta_1 = 1) = f(pt^{\circ}_{1,1}(t_0)+t_0)+g(pt^{\circ}_{1,1}(t_0))$, illustrated in figure 184, is provided by lemma 2. It is specified by the initial value 0.5, by the set { –21, –20, –13, –11, –8.5, 3, 7, 7.5, 9.5, 11, 12, 15, 17.25, 21 } of abscissae γ_i, $i = 1,\ldots,14$, at which the slope changes, and by the set { 1, 0, 1, 0.5, 1, 1.5, 2, 2.5, 3.5, 4.5, 6.5, 7.25, 6.25, 7.25 } of slopes μ_i, $i = 1,\ldots,14$, in the various intervals.

Case ii) in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable τ which corresponds to the processing time $pt_{2,1}$, the following function

$$\alpha_{2,1}\ \max\{t_0+st_{0,2}+\tau-dd_{2,1}\,,\,0\} + \beta_2\,(pt_2^{\mathrm{nom}}-\tau) + sc_{0,2} + J^{\circ}_{0,1,2}(t_1)$$

that can be written as $f(pt_{2,1}+t_0)+g(pt_{2,1})$ being

$$f(pt_{2,1}+t_0) = 2 \cdot \max\{pt_{2,1}+t_0-21\,,\,0\} + J^{\circ}_{0,1,2}(pt_{2,1}+t_0)$$

$$g(pt_{2,1}) = \begin{cases} 1.5\cdot(6-pt_{2,1}) & pt_{2,1} \in [4,6) \\ 0 & pt_{2,1} \notin [4,6) \end{cases}$$

The function $pt^{\circ}_{2,1}(t_0) = \arg\min_{pt_{2,1}}\{f(pt_{2,1}+t_0)+g(pt_{2,1})\}$, with $4 \le pt_{2,1} \le 6$, is determined by applying lemma 1. It is (see figure 183)

$$pt^{\circ}_{2,1}(t_0) = x_{\mathrm{e}}(t_0) \qquad \text{with} \quad x_{\mathrm{e}}(t_0) = \begin{cases} 6 & t_0 < 2.5 \\ -t_0+8.5 & 2.5 \le t_0 < 4.5 \\ 4 & t_0 \ge 4.5 \end{cases}$$

The conditioned cost-to-go $J^{\circ}_{0,0,0}(t_0 \mid \delta_2 = 1) = f(pt^{\circ}_{2,1}(t_0)+t_0)+g(pt^{\circ}_{2,1}(t_0))$, illustrated in figure 184, is provided by lemma 2. It is specified by the initial value 1, by the set { –13.5, –12, –11, –7.5, 2.5, 6.5, 7.5, 8.5, 10.5, 11, 11.5, 12.5, 13.5, 16, 17, 21.25, 26 } of abscissae γ_i, $i = 1,\ldots,17$, at which the slope changes, and by the set { 1, 0, 0.5, 1, 1.5, 1.75, 2.25, 3.25, 4.25, 3.25, 3.75, 5.25, 4.25, 5.25, 7.25, 6.25, 7.25 } of slopes μ_i, $i = 1,\ldots,17$, in the various intervals.

In order to find the optimal cost-to-go $J^{\circ}_{0,0,0}(t_0)$, it is necessary to carry out the following minimization

$$J^{\circ}_{0,0,0}(t_0) = \min\left\{J^{\circ}_{0,0,0}(t_0 \mid \delta_1 = 1)\,,\, J^{\circ}_{0,0,0}(t_0 \mid \delta_2 = 1)\right\}$$

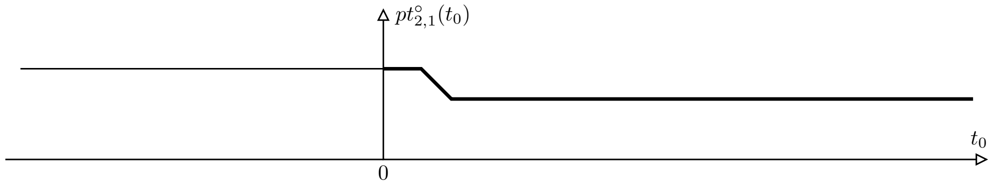


Figure 183: Optimal processing time $pt^\circ_{2,1}(t_0)$, under the assumption $\delta_2 = 1$ in the initial state $[0\ 0\ 0\ t_0]^T$.

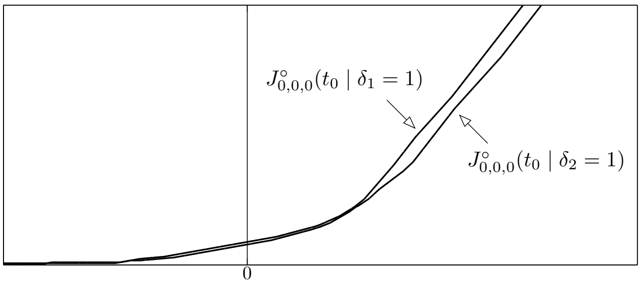


Figure 184: Conditioned costs-to-go $J^\circ_{0,0,0}(t_0 \mid \delta_1 = 1)$ and $J^\circ_{0,0,0}(t_0 \mid \delta_2 = 1)$ in the initial state $[0\ 0\ 0\ t_0]^T$.

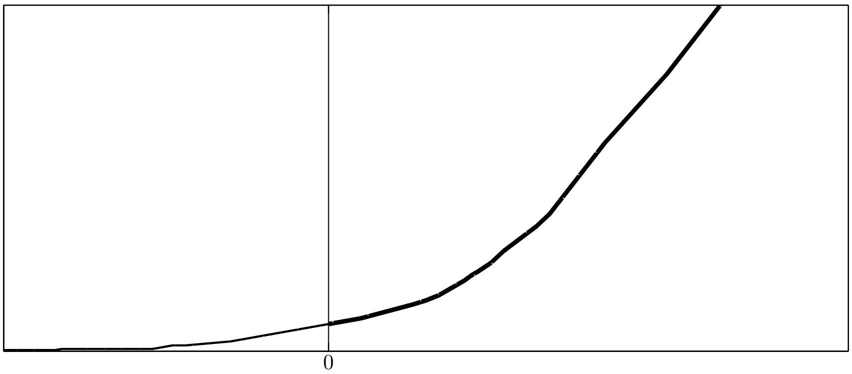


Figure 185: Optimal cost-to-go $J^\circ_{0,0,0}(t_0)$ in the initial state $[0\ 0\ 0\ t_0]^T$.

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 185.

The function $J^\circ_{0,0,0}(t_0)$ is specified by the initial value 0.5, by the set { –21, –20.5, –13.5, –12, –11, –7.5, 2.5, 6.5, 7.5, 8.5, 10.5, 11, 11.5, 12.5, 13.5, 16, 17, 21.25, 26 } of abscissae γ_i, $i = 1, \ldots, 19$, at which the slope changes, and by the set { 1, 0, 1, 0, 0.5, 1, 1.5, 1.75, 2.25, 3.25, 4.25, 3.25, 3.75, 5.25, 4.25, 5.25, 7.25, 6.25, 7.25 } of slopes μ_i, $i = 1, \ldots, 19$, in the various intervals.

Since $J^\circ_{0,0,0}(t_0 \mid \delta_1 = 1)$ is the minimum in $(-\infty, -20.5)$, and $J^\circ_{0,0,0}(t_0 \mid \delta_2 = 1)$ is the minimum in $[-20.5, +\infty)$ (see again figure 184), the optimal control strategies for the initial state are

$$\delta_1^\circ(0,0,0,t_0) = \begin{cases} 1 & t_0 < -20.5 \\ 0 & t_0 \geq -20.5 \end{cases} \qquad \delta_2^\circ(0,0,0,t_0) = \begin{cases} 0 & t_0 < -20.5 \\ 1 & t_0 \geq -20.5 \end{cases}$$

$$\tau^\circ(0,0,0,t_0) = \begin{cases} 8 & t_0 < -21 \\ -t_0 - 13 & -21 \leq t_0 < -20.5 \\ 6 & -20.5 \leq t_0 < 2.5 \\ -t_0 + 8.5 & 2.5 \leq t_0 < 4.5 \\ 4 & t_0 \geq 4.5 \end{cases}$$

The optimal control strategy $\tau^\circ(0,0,0,t_0)$ is illustrated in figure 186.

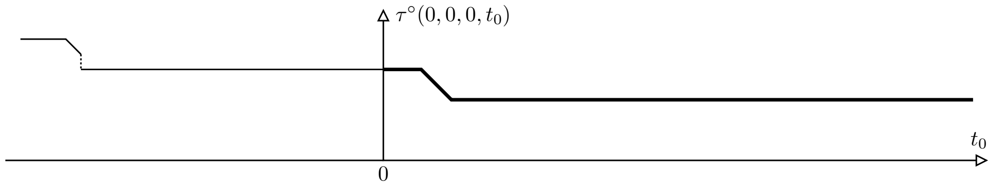


Figure 186: Optimal control strategy $\tau^\circ(0,0,0,t_0)$ in the initial state $[0\ 0\ 0\ t_0]^T$.

References

[1] Michele Aicardi, Davide Giglio, and Riccardo Minciardi. Optimal strategies for multiclass job scheduling on a single machine with controllable processing times. *IEEE Transactions on Automatic Control*, 53(2):479–495, March 2008.

[2] Davide Giglio. Optimal control strategies for single machine family scheduling with sequence-dependent batch setup and controllable processing times. *Journal of Scheduling*. Under review.